Astronomy 7A Final Exam

December 14, 2012

Name: ______________________________

Section: ______________________________

There are 6 problems of varying length that are required, and 1 bonus problem that is optional.

Write your answers on these sheets showing all of your work. It is better to show some work without an answer than to give an answer without any work. Feel free to use the back of the pages as well, but please clearly label which work corresponds to which problem.

Calculators are allowed to perform arithmetic. Please turn off all cellphones.

If you have any questions while taking the exam, get the attention of one of the GSIs or the instructor.

Budget your time; you will have from 9:30 am to 12:30 pm to complete the exam. Of course, you are free to hand in your exam before 12:30 pm. Make sure that you have time to at least briefly think about every required question.

You do not need to work on the questions in order, so it is OK to skip a question and come back to it later.

Constants

c $= 3.00 \times 10^{10}$ cm/s $= 3.00 \times 10^{8}$ m/s
h $= 6.623 \times 10^{-27}$ erg s $= 6.623 \times 10^{-34}$ J s
$\sigma_{\rm SB} = 5.67 \times 10^{-5}$ erg s^{-1} cm^{-2} K^{-4} $= 5.67 \times 10^{-8}$ W m^{-2} K^{-4}
$a = 4\sigma_{\rm SB}/c = 7.57 \times 10^{-15}$ erg s^{-1} cm^{-3} K^{-4} $= 7.57 \times 10^{-16}$ J s^{-1} m^{-3} K^{-4}
$k = 1.38 \times 10^{-16}$ erg/K $= 1.38 \times 10^{-23}$ J/K $= 8.62 \times 10^{-5}$ eV/K
$G = 6.67 \times 10^{-8}$ dyne cm^2/g^2 $= 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}
1 dyne = 1 g cm/s^2 $= 10^{-5}$ N
$m_p = 1.673 \times 10^{-24}$ g $= 1.673 \times 10^{-27}$ kg
$m_n = 1.675 \times 10^{-24}$ g $= 1.675 \times 10^{-27}$ kg
$m_e = 9.11 \times 10^{-28}$ g $= 9.11 \times 10^{-31}$ kg
$L_\odot = 3.90 \times 10^{33}$ erg/s $= 3.90 \times 10^{26}$ W
$M_\odot = 2.0 \times 10^{33}$ g $= 2.0 \times 10^{30}$ kg
$R_\odot = 7.0 \times 10^{10}$ cm $= 7.0 \times 10^{8}$ m
$T_\odot \approx 5800$ K (surface temperature)
1 AU $= 1.5 \times 10^{13}$ cm $= 1.5 \times 10^{11}$ m
1 pc $= 3.09 \times 10^{18}$ cm $= 3.09 \times 10^{16}$ m
1 eV $= 1.6 \times 10^{-12}$ erg $= 1.6 \times 10^{-19}$ J
1 radian = 206265 arcsec
1 Angstrom $= 10^{-8}$ cm $= 10^{-10}$ m
1 year = 12 months = 365 days $= 3.15 \times 10^{7}$ s

Some Useful Formulae

Kepler's Third Law:

$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)} a^3$$

For the relative 2-body orbit ($\equiv$ means "defined as"):

$$\vec{r} \equiv \vec{r}_2 - \vec{r}_1$$

$$\vec{v} \equiv \dot{\vec{r}}$$

$$r = \frac{a(1 - e^2)}{1 + e\cos(\theta - \beta)}$$

$$\vec{h} \equiv \vec{r} \times \vec{v}$$

$$h = \sqrt{a(1 - e^2)G(m_1 + m_2)}$$

$$E = -\frac{Gm_1 m_2}{2a}$$

For the orbit of m_2 relative to the barycenter:

$$\vec{r}_2 = \frac{m_1}{m_1 + m_2} \vec{r}$$

For the orbit of m_1 relative to the barycenter:

$$\vec{r}_1 = -\frac{m_2}{m_1 + m_2} \vec{r}$$

Virial Theorem:

$$\langle U \rangle = -2\langle K \rangle$$

Mass Function:

$$f \equiv \frac{m_2^3}{(m_2 + m_1)^2} \sin^3 i = \frac{4\pi^2 (a_1 \sin i)^3}{GP^2}$$

Distance in parsec, in terms of parallactic angle in arcsec:

$$D\,(\text{in pc}) = 1/p^{''}$$

Classical Doppler Shift ($v_r \ll c$):

$$\frac{(\lambda_{\text{observed}} - \lambda_{\text{rest}})}{\lambda_{\text{rest}}} = v_r/c$$

Blackbody *Surface* Flux:

$$F_{\text{blackbody}} = \sigma_{\text{SB}} T^4$$

Blackbody Radiation Energy Density (in *isotropic* radiation field):

$$u_{\text{blackbody}} = aT^4$$

Blackbody Radiation Pressure (in *isotropic* radiation field):

$$P_{\text{blackbody}} = u_{\text{blackbody}}/3$$

Planck Function:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/(\lambda kT)} - 1}$$

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/(kT)} - 1}$$

Wien Peak Law for B_λ:

$$h\nu_{\text{peak}} \approx 5kT$$

Wien Peak Law for B_ν:

$$h\nu_{\text{peak}} \approx 3kT$$

Diffraction Limit:

$$\theta_{\text{diff}} \approx \lambda/D$$

Energy Levels of Hydrogen:

$$E_n = -13.6 \text{ eV}/n^2$$

Degeneracy of Neutral Hydrogen:

$$g_n = 2n^2$$

Maxwell-Boltzmann Distribution of Velocities:

$$dn/dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/(2kT)} 4\pi v^2$$

Ideal Gas Law:

$$P = nkT = \rho kT/(\mu m_{\mathrm{H}})$$

Relative Boltzmann Probabilities:

$$\frac{N_b}{N_a} = \frac{g_b}{g_a} e^{-(E_b - E_a)/kT}$$

Partition Function For Use in Saha Equation:

$$Z = \sum_i g_i e^{-(E_i - E_1)/(kT)}$$

Saha Equation (for state I vs. II; but generalizable):

$$\frac{n_{\mathrm{II}} n_e}{n_{\mathrm{I}}} = \frac{2Z_{\mathrm{II}}}{Z_{\mathrm{I}}} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_1/kT}$$

Energy-Momentum for Photons
(also approximately true for massive relativistic particles):

$$E = cp$$

Optical Depth:

$$\tau = \int n\sigma dx$$

Mean Free Path:

$$\lambda_{\mathrm{mfp}} = 1/(n\sigma) = 1/(\rho\kappa)$$

Hydrostatic Equilibrium:

$$\frac{dP}{dr} = -\rho g$$

De Broglie Wavelength:

$$\lambda = h/p$$

For Ideal Gases That Behave Adiabatically:

$$P \propto \rho^\gamma$$

Adiabatic Temperature Gradient:

$$\frac{dT}{dr} = -\frac{\gamma - 1}{\gamma}\frac{g\mu m_{\mathrm{H}}}{k}$$

Energy released per pp reaction: 27 MeV.

1 A Star and a Planet

[2 points] (a) A star is observed to have an annual parallax motion of 0.1 arcseconds as seen from the Earth. **How far away is the star from the Earth? Give your answer in either AU or pc.**

1 arcsecond and 1 AU gives 1 pc. Baseline of 2 AU (annual motion) and 0.1 arcsecon gives $\boxed{20 \text{ pc}}$.

[2 points] (b) An extrasolar planet is observed to orbit this star. The planet's orbit is observed perfectly edge-on. The periastron distance on the sky is 1 *arcsecond. The apastron distance on the sky is* 3 *arcsecond.*

What is the eccentricity of the orbit?

Periastron distance is $a(1-e) = 1$. Apastron distance is $a(1+e) = 3$. Divide the two relations to find $(1-e)/(1+e) = 1/3$ which means $\boxed{e = 0.5}$.

[2 points] (c) The orbital period of the planet is 40 years. **What is the mass of the star, in solar masses $M_\odot$?** *Assume that the planet's mass is negligible compared to the star's mass.*

From (a) the distance is 20 pc, and from (b) we are told that periastron is 1 arcsecond. So that means the periastron physical distance is 20 AU. Also from (b), we know the eccentricity is 0.5, so that means $a = 20\text{AU}/(1-e) = 20\text{AU}/0.5 = 40$ AU.

We know from Kepler's Third Law that period $P \propto a^{3/2}/M^{1/2}$. Then $M \propto a^3/P^2$. Working in units of solar masses, years, and AU, we find that $M = 1M_\odot \times (40\text{AU})^3/(40\text{yr})^2 = \boxed{40M_\odot}$.

[2 points] (d) The star's flux as measured on Earth is F. The distance from the Earth to the star is d. The distance from the star to the planet is a.

Imagine yourself standing on the extrasolar planet with the star shining directly overhead. **Assuming the ground beneath you is a blackbody, write down a symbolic expression for the temperature of the ground, in terms of the variables given and fundamental constants.**

The intrinsic luminosity of the star is $L = 4\pi d^2 F$. The temperature of the planet is given by $\sigma T^4 = L/(4\pi a^2)$, so $\boxed{T = (F/\sigma)^{1/4}(d/a)^{1/2}}$. Any answer that gets to within a factor of 2 of this is correct.

2 Degeneracy

[8 points] Consider a stellar core near the end of its hydrogen-burning life. The core is about to run out of hydrogen and will start to collapse. Before the core shrinks, both the free electrons and the nuclei (ions) are ideal gases at the same temperature T. Assume both the electrons and the nuclei are non-relativistic throughout this problem.

As the core shrinks, the electrons become degenerate BEFORE the ions do. Explain why using physics and mathematics.

You may answer this question by considering the ratio between the packing parameter of the electrons and the packing parameter of the ions.

A star is said to be degenerate when the packing parameter is about 1, i.e. $\mathcal{P} \equiv n\lambda^3 \approx 1$. Here, λ is De Broglie wavelength, $\lambda = h/p$, where p is momentum, $p = mv \approx m\sqrt{\frac{kT}{m}} = \sqrt{mkT}$. So,

$$\mathcal{P} = \frac{nh^3}{(mkT)^{3/2}}$$

The ratio between the packing parameter of the electrons and the packing parameter of the ions is then

$$\frac{\mathcal{P}_e}{\mathcal{P}_p} = \frac{\frac{h^3 n_e}{(m_e kT)^{3/2}}}{\frac{h^3 n_{\text{ion}}}{(m_{\text{ion}} kT)^{3/2}}}.$$

The ions for this problem are Helium ions. So there are 2 electrons for every Helium (charge = +2) ion. Thus $n_e/n_{\text{ion}} = 2$, and

$$\boxed{\frac{\mathcal{P}_e}{\mathcal{P}_p} = 2\left(\frac{m_{\text{ion}}}{m_e}\right)^{3/2}}$$

Now an electron is about 1800×4 times lighter than a helium ion. Therefore when $\mathcal{P}_e \approx 1$—i.e., when the electrons are degenerate—that implies $\mathcal{P}_p \approx 0.5/(1800 \times 4)^{3/2} \ll 1$—i.e., the helium nuclei are not packed yet (the helium nuclei are not degenerate yet). Therefore, $\boxed{\text{the electrons become degenerate } \textit{before} \text{ the hel}}$

3 The Alpha, Beta, and Gammas of Stars

Consider stars of mass M and radius R in which particles scatter photons with a cross-section

$$\sigma \propto \rho^\alpha T^\beta \tag{1}$$

where ρ is the mass density and T is the temperature. Assume the constants α and β are known. *Note: Do not confuse the above cross section σ with the Stefan-Boltzmann constant σ_{SB} in the formula for blackbody flux!*

The gas is heated by nuclear fusion in the star's core. Energy is transported from the core to the surface radiatively, by photons.

Use the approximation that the central temperature is constant with stellar mass ($T_c \propto M^0$).

(a) [4 points] The star is supported against gravity by gas pressure.

Show that the central temperature of the star can be approximated as $T_\mathrm{c} \sim GM\mu m_\mathrm{H}/(kR)$, where G is the gravitational constant, μ is the mean molecular weight, m_H is the mass of the hydrogen atom, and k is the Boltzmann constant. *Every step of your derivation should be clearly stated.*

The star is in hydrostatic equilibrium, so we start with $\frac{dP}{dr} = -\rho g$. To order of magnitude,

$$\frac{dP}{dr} \sim \frac{\Delta P}{\Delta r} = \frac{P_c - P_s}{0 - R} = -\frac{P_c}{R}$$

where P_c is the pressure at the center and P_s is the pressure at the surface (which is much, much smaller than P_c).

Also, g can be written as GM/R^2 from Newton's Law of Gravity. So,

$$\begin{aligned} -\frac{P_c}{R} &\sim -\rho\frac{GM}{R^2} \\ P_c &\sim \rho\frac{GM}{R} \end{aligned} \tag{2}$$

We are told that this star is supported by gas pressure, so $P_c \approx P_\mathrm{gas,center} = \frac{\rho}{\mu m_H}kT_c$. By sticking this equation of state and the hydrostatic equilibrium equation together,

$$\begin{aligned} \frac{\rho}{\mu m_H}kT_c &\sim \rho\frac{GM}{R} \\ T_c &\sim \frac{GM\mu m_H}{kR} \end{aligned} \tag{3}$$

(b) For parts (b) and (c) of this problem, you may use $T_\mathrm{c} \sim GM\mu m_\mathrm{H}/(kR)$.

[4 points] Derive how the central pressure P_c scales with stellar mass M. A proportionality is sufficient. Every step of your derivation should be clearly stated.

If you use certain equations or proportionalities that you have memorized, you should explain why your equations are true, or where they come from.

From $T_c \sim GM\,\mu \mathrm{m}_H/(kR)$, we have $T_c \propto M/R$. Using the assumption, $T_c \propto M^0$, $T_c \propto \frac{M}{R} \propto M^0$. Thus, we get $\boxed{M \propto R}$.

Then we apply this proportionality to the hydrostatic equilibrium equation (along with the fact that $\rho \propto M/R^3$) to get

$$\frac{P_c}{R} \sim \rho\frac{GM}{R^2} \propto \frac{M}{R^3}\frac{M}{R^2} \propto \frac{M^2}{R^5} \tag{4}$$

or $\boxed{P_c \propto M^2/R^4 \propto M^{-2}}$

(c) [6 points] Derive how the stellar luminosity L scales with surface effective temperature T_{eff}. A proportionality is sufficient. You may use whatever relations are relevant from parts (a) and (b). Every step of your derivation should be clearly stated.

Let's start with the radiative transport equation, $\frac{dT}{dr} = -\frac{3\rho\kappa}{4acT^3}\frac{L(r)}{4\pi r^2}$. To order of magnitude,

$$\frac{dT}{dr} \sim \frac{\Delta T}{\Delta r} = \frac{T_c - T_s}{0 - R} = -\frac{T_c}{R}$$

where T_s is the surface temperature (which is much, much smaller than T_c). Also, $L(r)/(4\pi r^2)$ is approximated as $L/(4\pi R^2) \propto L/R^2$.

κ is the cross-section per unit mass, so $\kappa \propto \sigma/(\mu m_H) \propto \rho^\alpha T^\beta \propto \rho^\alpha T_c^\beta$.

Plugging all this back into the equation of radiative transport and dropping the constants (since we only care about a proportionality) and using the relation $M \propto R$ that we got in (b) and our friend $\rho \propto M/R^3$,

$$\begin{aligned}
-\frac{T_c}{R} &\propto -\rho\,\kappa\,T_c^{-3}\,\frac{L}{R^2} \\
\frac{T_c}{R} &\propto \rho\,(\rho^\alpha\,T_c^\beta)\,T_c^{-3}\frac{L}{R^2} \propto \left(\frac{M}{R^3}\right)^{(\alpha+1)} T_c^{(\beta-3)}\frac{L}{R^2} \\
T_c &\propto M^{(\alpha+1)}\,R^{(-3\alpha-4)}\,T_c^{(\beta-3)}\,L \\
T_c &\propto M^{(-2\alpha-3)}\,T_c^{(\beta-3)}\,L
\end{aligned} \tag{5}$$

Rearranging the above equation and using the assumption $T_c \propto M^0$, we get $\boxed{L \propto M^{(2\alpha+3)}\, T_c^{(-\beta+4)} \propto M^{(2\alpha+3)}}$.

Combining the boxed proportionality above with the Stefan-Boltzmann equation, $L = 4\pi R^2 \sigma_{\rm SB} T_{\rm eff}^4 \propto R^2 T_{\rm eff}^4$, and using the mass-radius relation from part (b), $M \propto R$,

$$\begin{aligned} L \propto R^2 T_{\rm eff}^4 &\propto M^{(2\alpha+3)} \\ M^2 T_{\rm eff}^4 &\propto M^{(2\alpha+3)} \\ M^{(2\alpha+1)} &\propto T_{\rm eff}^4 \\ M &\propto T_{\rm eff}^{4/(2\alpha+1)} \end{aligned} \tag{6}$$

Plugging the above effective temperature-mass relation back in to the Stefan-Boltzmann equation gives

$$L \propto R^2 T_{\rm eff}^4 \propto M^2 T_{\rm eff}^4$$

$$L \propto \left(T_{\rm eff}^{4/(2\alpha+1)}\right)^2 T_{\rm eff}^4 \propto T_{\rm eff}^{8/(2\alpha+1)} T_{\rm eff}^4$$

or $\boxed{L \propto T_{\rm eff}^{(8\alpha+12)/(2\alpha+1)}}$

4 Supernova

Consider a star about to go supernova. The iron core of mass $M_{\rm core}$ and radius $R_{\rm core}$ is about to collapse. The iron core is surrounded by the rest of the star, of mass M.

The bulk of the energy released in the supernova derives from the gravitational energy released by the collapse of the core into a neutron star, of radius $R_{\rm NS}$. Call this energy E.

A fraction f of the total energy released is imparted to the kinetic energy of the supernova remnant.

(a) [4 points] Write down a rough symbolic expression for the velocity v of gas in the supernova remnant, in terms of the variables given.

The total energy released comes from the change in potential energy as the core collapses and is given by

$$E = \frac{GM_{\text{core}}^2}{R_{\text{NS}}} - \frac{GM_{\text{core}}^2}{R_{\text{core}}} \approx \frac{GM_{\text{core}}^2}{R_{\text{NS}}}$$

since the radius of a neutron star is much smaller than the radius of the core (effectively the radius of a white dwarf). The kinetic energy of the supernova remnant is $Mv^2/2$. So,

$$\begin{aligned} fE &= \frac{1}{2}Mv^2 \\ v &= \sqrt{\frac{2fE}{M}} \end{aligned} \tag{7}$$

You could have also written this as

$$v = \sqrt{\frac{2fGM_{\text{core}}^2}{R_{\text{NS}}M}}$$

Either version received full credit.

(b) [4 points] Give a numerical estimate for v, in km/s. You may assume $M \approx 20M_\odot$ and $f \sim 1\%$, but the remaining variables you will have to draw from your knowledge of astronomy.

From the result of (a),

$$v = \sqrt{\frac{2f}{M}\left(\frac{GM_{\text{core}}^2}{R_{\text{NS}}} - \frac{GM_{\text{core}}^2}{R_{\text{core}}}\right)} \approx \sqrt{\frac{2f}{M}\frac{GM_{\text{core}}^2}{R_{\text{NS}}}}$$

again assuming $R_{\text{NS}} \ll R_{\text{core}}$. Plugging in typical numbers, $R_{\text{NS}} \approx 10\text{km}$, $M_{\text{core}} \approx 2M_\odot$, we get approximately $\boxed{7300 \text{ km/s}}$.

5 Astrometric Binary

Two stars are observed to execute the following motion on the sky:

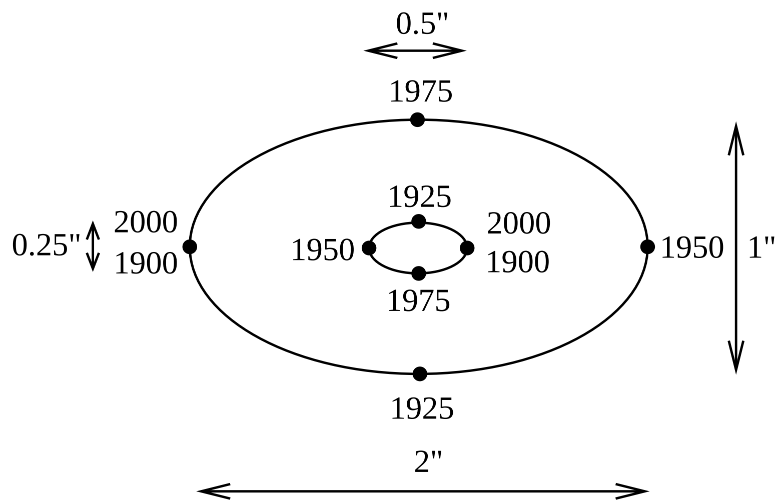

Figure 1: Astrometric motion of two stars.

where the positions are labeled by the year of observation, and the dimensions of the orbits are measured in arcseconds.

The outer, larger orbit is occupied by Star 2. The inner, smaller orbit is occupied by Star 1.

(a) [2 points] The orbit of either star about the center of mass is actually perfectly circular (as can be deduced from Figure 1). **From the fact that the orbital eccentricity is zero, and using Figure 1, calculate the inclination of the orbit relative to the sky plane.**

90 degrees minus the inverse sine of 1/2, or $\boxed{\text{60 degrees}}$ relative to the sky plane.

(b) [6 points] In 2000, a spectrum of Star 1 (the one with the smaller orbit) is taken. The Hα line, having a rest wavelength of 6562.81

Angstroms, is observed instead to have a wavelength of 6562.81 + 0.0113 Angstroms. **Calculate precisely the semi-major axis a_1 of the orbit of Star 1 about the center of mass, in AUs.**

Ignore any motion the center of mass may have.

We use the Doppler formula to convert the wavelength shift of $\Delta\lambda = 0.0113$ Angstroms into a RADIAL (parallel to the line of sight) velocity:

$$\frac{\Delta\lambda}{\lambda} = \frac{v_r}{c} \tag{8}$$

which implies that $v_r = 5.16 \times 10^4$ cm/s. We have used $\lambda = 6562.81$ Angstroms and $c = 3 \times 10^{10}$ cm/s.

Now the crucial thing to realize is that this velocity is NOT the full orbital velocity v_1 of the star. That's because the orbit is seen at an angle. The observed Doppler shift is only sensitive to a component of the full orbital velocity. That is, $v_r = v_1 \sin i$, where i is the inclination angle of the orbit ($i = 0$ for face-on orbits, and $i = 90$ deg for edge-on orbits).

What is i for this orbit? We can deduce this from the figure. We know the orbit of the star is circular but the observed axis ratio is 1/2, which means the inclination angle is 60° and $\sin i = \sqrt{3}/2$.

So $v_1 = v_r / \sin i = v_r/(\sqrt{3}/2)$. Now we can plug into

$$v_1 = \frac{2\pi a_1}{P} \tag{9}$$

where $P = 100$ yr, and solve for $\boxed{a_1 = 2 \text{ AU}}$.

(c) [6 points] **Calculate M_1 (the mass of Star 1) and M_2 (the mass of Star 2), in solar masses.** *Are these brown dwarfs, stars, or compact objects (white dwarfs, neutron stars, black holes)? Or is it impossible to tell from the information given?*

We start with Kepler's 3rd law,

$$P^2 = \frac{4\pi^2}{G(M_1 + M_2)}(a_1 + a_2)^3 \tag{10}$$

From Figure 1, we know the orbits are circular and that $a_1/a_2 = 1/4$. So because $a_1 = 2$ AU from part (b), it must be that $a_2 = 2$ AU $\times 4 = 8$ AU, and $a_1 + a_2 = 10$ AU. We insert this into Kepler III and solve for $M_1 + M_2 = 0.1 M_\odot$.

Now we use

$$M_1 a_1 = M_2 a_2 \tag{11}$$

which is just a statement about the center of mass. From this we know $M_1/M_2 = a_2/a_1 = 4/1$. So knowing $M_1 + M_2 = 0.1 M_\odot$ and knowing $M_1/M_2 = 4/1$, we solve for $\boxed{M_1 = 0.08 M_\odot}$ and $\boxed{M_2 = 0.02 M_\odot}$. These must be $\boxed{\text{brown dwarfs}}$ because their masses are $\leq 0.08 M_\odot$, below the hydrogen-burning limit. If you said a brown dwarf and a star, you would get full credit.

6 Main Sequence Dating

The following Hertzsprung-Russell diagram contains data from two separate stellar clusters (one represented by squares; the other by circles):

(a) [4 points] About what age is the cluster which is represented by squares? Give your answer in Gyr. Your answer should be good to within a factor of two.

Since all the stars in the cluster were born at the same time, the age of the cluster must equal the Main Sequence lifetime of the stars that are just beginning to leave the Main Sequence. The cluster represented by squares has stars that are about one solar luminosity just starting to leave the Main

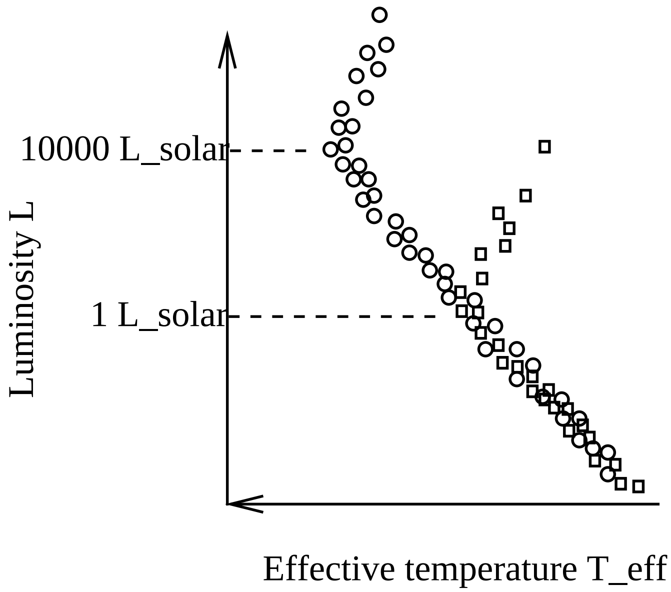

Figure 2: HR diagram showing two clusters: one in squares, and the other in circles.

Sequence. We know that these stars must be about one solar mass, so the age of the cluster is equal to the Main Sequence lifetime of a one solar mass star. One solar mass stars on the Main Sequence have hydrogen fusion dominated by the pp-chain. So,

$$t_{\rm MS} \sim \frac{0.1 M_\odot c^2 f_{pp}}{L_\odot}, \text{ where } f_{pp} = \frac{27\text{MeV}}{4 m_p c^2} \approx 0.007$$

Note that the 0.1 comes from the fact that Main Sequence hydrogen fusion occurs only in the inner 10% of the star (i.e., the core). Plugging in the numbers, we get about $\boxed{\text{10 Gyrs}}$. You also could have just stated that the Main Sequence lifetime of the Sun is about 10 Gyr if you happened to know that number. Also note that 1 Gyr is equal to 10^9 years.

(b) [6 points] Assume that while on the main-sequence, the stellar luminosity L scales with stellar mass M as $L \propto M^4$ (actually this scaling is not accurate for moderate mass stars, but the numbers work out nicely if we use this scaling anyway. What is important is the reasoning behind your answer).

About what age is the cluster which is represented by circles? Give your answer in Gyr.

The life time on main sequence,

$$t_{\rm MS} \propto \frac{E}{L} \propto \frac{M}{L} \propto \frac{M}{M^4} \propto M^{-3} \propto L^{-3/4}$$

Taking the ratio of the age of the stars represented by circles (with $L = 10^4 L_\odot$) to the age of the stars represented by squares (with $L = L_\odot$),

$$\frac{t_{\rm circle}}{t_{\rm square}} \approx \left(\frac{10^3 L_\odot}{L_\odot}\right)^{-3/4} \approx 10^{-3}$$

From part (a) we know $t_{\rm square} = 10$ Gyr, so the age of the circle cluster is about $\boxed{0.01 \text{ Gyr}} = 10$ Myr.

7 Bonus Question (OPTIONAL)

[6 points] Consider a fully convective M dwarf having a mass $M = 0.2M_\odot$, a radius $R = 0.2R_\odot$, and a surface temperature of $T_{\rm s} = 2000$ K.

Estimate the surface pressure $P_{\rm s}$ of the M dwarf.

The numbers given are not enough to solve this problem. You will have to draw from your knowledge of physics and astronomy.

Because the star is fully convective, we know the adiabatic relation $P \propto \rho^\gamma$ must hold approximately throughout the star. Combine this with the ideal gas law $P \propto \rho T$ and we have $P \propto T^{\gamma/(\gamma-1)}$. For this star, $\gamma = 5/3$ (the gas is purely monatomic; actually it's fully ionized inside the star; in any case each gas particle has no rotational degrees of freedom), so we have $P \propto T^{5/2}$, which implies $P_s/P_c = (T_s/T_c)^{5/2}$. We know from hydrostatic equilibrium that $P_c \sim GM^2/R^4$ (see also problem 3b above) and M and R are given by the problem; we also know from class that $T_c \sim 10^7$ K; and we are given that $T_s \sim 2000$ K. So plugging it all in, we get $\boxed{P_s \sim 10^8 \text{ dyne/cm}^2}$. Any answer (justified by physical reasoning) within a factor of 10 of this received full credit.