Astronomy 7A Midterm #1

October 6, 2012

Name: ______________________________

Section: ______________________________

There are 4 problems of roughly equal weight that are required. In addition, there is a 5th optional bonus problem.

Write your answers on these sheets showing all of your work. It is better to show some work without an answer than to give an answer without any work. Feel free to use the back of the pages as well, but please clearly label which work corresponds to which problem.

Calculators are allowed to perform arithmetic. Please turn off all cellphones.

If you have any questions while taking the midterm, get the attention of one of the GSIs or the instructor.

Budget your time; you will have from 12:40 pm to 2:00 pm to complete the exam. Of course, you are free to hand in your exam before 2:00 pm. Make sure that you have time to at least briefly think about every required question on the midterm.

You do not need to work on the questions in order, so it is OK to skip a question and come back to it later.

Constants

c = 3.00×10^{10} cm/s = 3.00×10^{8} m/s
h = 6.623×10^{-27} erg s = 6.63×10^{-34} m^2 kg s^{-1}
$\sigma = 5.67 \times 10^{-5}$ erg s^{-1} cm^{-2} K^{-4} = 5.67×10^{-8} W m^{-2} K^{-4}
$k = 1.38 \times 10^{-16}$ erg/K = 1.38×10^{-23} J/K
$G = 6.67 \times 10^{-8}$ dyne cm^2/g^2 = 6.67×10^{-11} m^3 kg^{-1} s^{-2}
1 dyne = 1 g cm/s^2 = 10^{-5} N
$m_p = 1.673 \times 10^{-24}$ g = 1.673×10^{-27} kg
$m_n = 1.675 \times 10^{-24}$ g = 1.675×10^{-27} kg
$m_e = 9.11 \times 10^{-28}$ g = 9.11×10^{-31} kg
$L_\odot = 3.90 \times 10^{33}$ erg/s = 3.90×10^{26} W
$M_\odot = 2.0 \times 10^{33}$ g = 2.0×10^{30} kg
$R_\odot = 7.0 \times 10^{10}$ cm = 7.0×10^{8} m
$T_\odot \approx 5800$ K (surface temperature)
1 AU = 1.5×10^{13} cm = 1.5×10^{11} m
1 pc = 3.09×10^{18} cm = 3.09×10^{16} m
1 eV = 1.6×10^{-12} erg = 1.6×10^{-19} J
1 radian = 206265 arcsec
1 Angstrom = 10^{-8} cm = 10^{-10} m
1 year = 12 months = 365 days = 3.15×10^{7} s

Some Useful Formulae

Classical Doppler Shift ($v_r \ll c$):

$$\frac{(\lambda_{\rm observed} - \lambda_{\rm rest})}{\lambda_{\rm rest}} = v_r/c$$

Blackbody Surface Flux:

$$F_{\rm blackbody} = \sigma T^4$$

Planck Function:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/(\lambda kT)} - 1}$$

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(kT)} - 1}$$

Wien Peak Law for B_λ:

$$h\nu_{\rm peak} \approx 5kT$$

Wien Peak Law for B_ν:

$$h\nu_{\rm peak} \approx 3kT$$

Diffraction Limit:

$$\theta_{\rm diff} \approx \lambda/d$$

Energy Levels of Hydrogen:

$$E_n = -13.6 \text{ eV}/n^2$$

1 Measure for Measure

(a) [2 points] A telescope of aperture area ΔA collects photons having wavelengths between λ_1 and λ_2 from an astrophysical source. Within a time interval Δt, it collects a total energy ΔE.

Write down a symbolic expression for the flux density F_ν, *in terms of the variables given and fundamental constants.*

The problem asks for F_ν not F_λ. We need the frequency interval, not the wavelength interval. λ_1 corresponds to a frequency of c/λ_1 and λ_2 corresponds to a frequency of c/λ_2. Therefore the frequency interval is $c/\lambda_1 - c/\lambda_2$.

$$\boxed{F_\nu = \frac{\Delta E}{\Delta t \Delta A (c/\lambda_1 - c/\lambda_2)}}$$

(b) [2 points] Suppose this same source is angularly resolved. It looks like a uniformly bright, circular disk with angular radius θ. **Write down a symbolic expression for the specific intensity** I_λ, *in terms of the variables given and fundamental constants.*

Specific intensity is flux density divided by solid angle. The solid angle of the disk is $\pi\theta^2$. And now the problem asks for I_λ not I_ν, so we need the wavelength interval $\lambda_2 - \lambda_1$.

$$\boxed{I_\lambda = \frac{\Delta E}{\Delta t \Delta A (\lambda_2 - \lambda_1) \pi \theta^2}}$$

(c) [2 points] Suppose the distance d to this same source is known. **Write down a symbolic expression for the source luminosity** L, *in terms of the variables given and fundamental constants. Assume the source radiates isotropically. (This is not the bolometric luminosity, but just the luminosity between wavelengths λ_1 and λ_2.)*

For an isotropically emitting source, $L = 4\pi d^2 F$, and $F = \Delta E / (\Delta A \Delta t)$. So

$$\boxed{L = 4\pi d^2 \frac{\Delta E}{\Delta t \Delta A}}$$

(d) [2 points] Suppose the source is known to be a blackbody, and we measure its specific intensity at a SINGLE wavelength. From this single measurement, **can we calculate the temperature of the blackbody? Yes or No, with BRIEF explanation.**

$\boxed{\text{Yes}}$, because the specific intensity of a blackbody is given by the Planck function, and the Planck function (on the formula sheet) depends only on λ and T. We are told that we know the specific intensity at a certain wavelength, so the only thing that is not known is T. We have 1 equation in 1 unknown, so we can solve for T.

2 Blackbody Star

Idealize a star as a spherical blackbody of temperature T and radius R. A telescope observes the star at a distance d. (a) [2 points] **Write down a symbolic expression for the bolometric luminosity L emitted by the star** *in terms of the variables given and fundamental constants.*

$\boxed{L = 4\pi R^2 \sigma T^4}$

(b) [2 points] **Write down a symbolic expression for the bolometric flux F detected by the telescope** *in terms of the variables given and fundamental constants.*

$\boxed{F = \frac{L}{4\pi d^2} = \sigma T^4 R^2/d^2}$

(c) [4 points] For this part (c), use only the information in (c). Do not use information previous to (c).

It is claimed that the angular radius θ of the star can be estimated by measuring (1) the bolometric flux F, and (2) the photon frequency ν_* where the spectrum — as measured by the flux density F_ν — peaks. (In other words, ν_* is the frequency where F_ν is maximum).

Is this claim true? If true, give an expression for θ in terms of the variables given and fundamental constants. If false, describe what additional measurements would be required.

$\boxed{\text{The statement is TRUE: We CAN estimate the angular size of a star}}$. A star can be approximated as a black body. So, first by measuring the peak of the flux density, F_ν, and using Wien's law, ($h\nu_{\text{peak}} \approx 3kT$ for B_ν), we can deduce the temperature of the star. Then from Stefan-Boltzmann's law,

$$\begin{aligned} L &= 4\pi R^2 \sigma T^4 \\ \frac{L}{4\pi d^2} &= \frac{4\pi R^2}{4\pi d^2} \sigma T^4 \\ F &= \left(\frac{R}{d}\right)^2 \sigma T^4 \end{aligned} \tag{1}$$

where d is the distance to a star. By definition of the angular size, $R/d = \theta$,

$$\boxed{F = \theta^2 \sigma T^4}.$$

We measure the bolometric flux F and calculate T above, so we can obtain the angular size

$$\boxed{\theta = \left(\frac{F}{\sigma}\right)^{1/2} \left(\frac{3k}{h\nu_*}\right)^2}$$

3 Diffraction and Parallax

An astronomer on Earth measures stellar parallaxes over an entire year.

The astronomer's telescope is diffraction limited. The mirror diameter is D, and the observing wavelength is λ. Ignore the effects of the atmosphere.

(a) [4 points] Only stars that are close enough to Earth — within a distance r — have parallaxes large enough to measure.

Write down an expression for r *in terms of the variables given and known constants. Your answer should be accurate to better than a factor of 2.*

Assume the orbit of the Earth is circular.

For a star that is r away, the parallactic angular shift of the star is given by $\theta_{\text{parallax}} = 2\,\text{AU}/r$ (2 AU for the full diameter of the Earth's orbit).

If we had perfect angular resolution, we could resolve any value of θ_{parallax}, no matter how small. But our instrumentation is diffraction limited. We can't resolve angles bigger than $\theta_{\text{diff}} = 1.22\lambda/D$ (using Rayleigh's criterion for a circular aperture).

So we set $\theta_{\text{parallax}} = \theta_{\text{diff}}$ and solve for $\boxed{r = 2\,\text{AU}/\theta_{\text{diff}} = 1.64\,\text{AU} \times D/\lambda}$.

(b) [4 points] If the astronomer wishes to increase the total number of measurable parallaxes (= parallaxes large enough to measure) by a factor of 10^3 *(a thousandfold increase),* **what changes can they make to their observing equipment? Specify two ways and be quantitative.**

Assume the volume density of stars in our solar neighborhood (the number of stars per unit volume) is constant.

We live in a 3D universe, so to increase the number of stars accessible by a factor of 10^3, we just need to increase the distance out to which we make good measurements by a factor of 10 (because the volume of a sphere scales as radius^3). So we need to increase r by a factor of 10. We can do this either by $\boxed{\text{decreasing } \lambda \text{ by a factor of 10}}$, or $\boxed{\text{increasing } D \text{ by a factor of 10}}$ to improve our angular resolution by a factor of 10.

4 The Black Hole and the Star

Consider a star orbiting a supermassive black hole of mass M. Assume the star has negligible mass compared to the black hole.

The orbit of the star is circular. An astronomer observes the orbit edge-on. Assume the black hole does not move relative to the observer.

(a) [4 points] The star's atmosphere contains hydrogen which produces an absorption line at Balmer γ (otherwise known as "H-gamma"). **Calculate the rest wavelength $\lambda_{\rm rest}$ of this absorption line, in Angstroms**. *Just 1 significant figure is enough!*

"Balmer" means a transition that lands to $n_{\rm final} = 2$. "γ" means the transition starts 3 steps above $n_{\rm final} = 2$, or $n_{\rm start} = 5$. So the electron in the hydrogen atom is transitioning from $n_{\rm start} = 5$ to $n_{\rm final} = 2$.

Use the formula (on the equation sheet) for the energy levels of hydrogen to solve for the energy difference between $n_{\rm start} = 5$ and $n_{\rm final} = 2$.

$$\Delta E = -13.6\,\mathrm{eV}\left(\frac{1}{5^2} - \frac{1}{2^2}\right) = 2.856\,\mathrm{eV} \tag{2}$$

The question asks for wavelength. Energy is related to wavelength as follows:

$$\Delta E = h\nu = hc/\lambda\,. \tag{3}$$

Solve for $\lambda = hc/\Delta E$ or $\boxed{\lambda = 4350 \text{ Angstroms}}$ (after doing the unit conversions).

(b) [4 points] The Balmer γ line is observed to undergo periodic redshifts and blueshifts. The maximum redshift observed is $\Delta\lambda$ (i.e., the line is observed at a wavelength of $\lambda_{\rm rest}+\Delta\lambda$, where $\Delta\lambda \ll \lambda_{\rm rest}$).

Obtain a symbolic expression for the orbital radius r of the star around the black hole*, in terms of the variables given and fundamental constants.*

The redshift $\Delta\lambda$ obeys $\Delta\lambda/\lambda = v_r/c$, where v_r is the line-of-sight velocity. Because the orbit is observed edge-on, the maximum line-of-sight velocity is just the circular orbital velocity of the star. So we need to solve for the orbital velocity.

We know that for circular orbits, $v = \sqrt{GM/r}$, so $r = GM/v^2$.

We just need v, which we get using the Doppler shift formula: $v = c\Delta\lambda/\lambda_{\rm rest}$. This v equals the radial velocity v_r because we are viewing the orbit edge-on.

So combining the two formulae, we get

$\boxed{r = GM\left(\frac{\lambda_{\rm rest}}{c\Delta\lambda}\right)^2}$

5 BONUS: Optical Depth

This problem is optional, but can be solved for bonus points.

[4 points] A slab of gas has optical depth τ.

A photon tries to penetrate the slab.

Write down the probability that the photon will be absorbed, in terms of τ and fundamental constants. Your expression should be valid for ANY value of τ.

Hint: You can start with the probability that the photon will pass through. Consider how flux attenuates (i.e., how flux decreases with increasing τ).

Flux attenuates as $F = F_0 \exp(-\tau)$, so the fraction of photons that pass through is $F/F_0 = \exp(-\tau)$. This equals the probability that a single photon passes through.

The probability that a photon does not pass through (gets absorbed) is one minus the probability that it passes through (so the sum of probabilities is one). Therefore the desired probability of absorption is $\boxed{1 - \exp(-\tau)}$.