Astronomy 7A Midterm #2

November 8, 2012

Name: ________________________________

Section: ________________________________

There are 5 problems that are required — and 1 bonus problem that is optional and that can *only improve* your letter grade.

Write your answers on these sheets showing all of your work. It is better to show some work without an answer than to give an answer without any work.

Calculators are allowed to perform arithmetic. Please turn off all cellphones.

If you have any questions while taking the midterm, get the attention of one of the GSIs or the instructor.

Budget your time; you will have from 12:40 pm to 2:00 pm to complete the exam. Of course, you are free to hand in your exam before 2:00 pm. Make sure that you have time to at least briefly think about every required question on the midterm.

You do not need to work on the questions in order, so it is OK to skip a question and come back to it later.

Constants

c $= 3.00 \times 10^{10}$ cm/s $= 3.00 \times 10^{8}$ m/s
h $= 6.623 \times 10^{-27}$ erg s $= 6.623 \times 10^{-34}$ J s
$\sigma = 5.67 \times 10^{-5}$ erg s^{-1} cm^{-2} K^{-4} $= 5.67 \times 10^{-8}$ W m^{-2} K^{-4}
$a = 4\sigma/c = 7.57 \times 10^{-15}$ erg s^{-1} cm^{-3} K^{-4} $= 7.57 \times 10^{-16}$ J s^{-1} m^{-3} K^{-4}
$k = 1.38 \times 10^{-16}$ erg/K $= 1.38 \times 10^{-23}$ J/K $= 8.62 \times 10^{-5}$ eV/K
$G = 6.67 \times 10^{-8}$ dyne cm^2/g^2 $= 6.67 \times 10^{-11}$ m^3 kg^{-1} s^{-2}
1 dyne = 1 g cm/s^2 $= 10^{-5}$ N
$m_p = 1.673 \times 10^{-24}$ g $= 1.673 \times 10^{-27}$ kg
$m_n = 1.675 \times 10^{-24}$ g $= 1.675 \times 10^{-27}$ kg
$m_e = 9.11 \times 10^{-28}$ g $= 9.11 \times 10^{-31}$ kg
$L_\odot = 3.90 \times 10^{33}$ erg/s $= 3.90 \times 10^{26}$ W
$M_\odot = 2.0 \times 10^{33}$ g $= 2.0 \times 10^{30}$ kg
$R_\odot = 7.0 \times 10^{10}$ cm $= 7.0 \times 10^{8}$ m
$T_\odot \approx 5800$ K (surface temperature)
1 AU $= 1.5 \times 10^{13}$ cm $= 1.5 \times 10^{11}$ m
1 pc $= 3.09 \times 10^{18}$ cm $= 3.09 \times 10^{16}$ m
1 eV $= 1.6 \times 10^{-12}$ erg $= 1.6 \times 10^{-19}$ J
1 radian = 206265 arcsec
1 Angstrom $= 10^{-8}$ cm $= 10^{-10}$ m
1 year = 12 months = 365 days $= 3.15 \times 10^{7}$ s

Some Useful Formulae

Kepler's Third Law:

$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)} a^3$$

For the relative 2-body orbit ($\equiv$ means "defined as"):

$$\vec{r} \equiv \vec{r}_2 - \vec{r}_1$$

$$\vec{v} \equiv \dot{\vec{r}}$$

$$r = \frac{a(1 - e^2)}{1 + e\cos(\theta - \beta)}$$

$$\vec{h} \equiv \vec{r} \times \vec{v}$$

$$h = \sqrt{a(1 - e^2)G(m_1 + m_2)}$$

$$E = -\frac{G m_1 m_2}{2a}$$

For the orbit of m_2 relative to the center-of-mass:

$$\vec{r}_2 = \frac{m_1}{m_1 + m_2}\vec{r}$$

For the orbit of m_1 relative to the center-of-mass:

$$\vec{r}_1 = -\frac{m_2}{m_1 + m_2}\vec{r}$$

Virial Theorem:

$$\langle U \rangle = -2\langle K \rangle$$

Mass Function:

$$f \equiv \frac{m_2^3}{(m_2 + m_1)^2}\sin^3 i = \frac{4\pi^2 (a_1 \sin i)^3}{G P^2}$$

Classical Doppler Shift ($v_r \ll c$):

$$\frac{(\lambda_{\mathrm{observed}} - \lambda_{\mathrm{rest}})}{\lambda_{\mathrm{rest}}} = v_r/c$$

Blackbody *Surface* Flux:

$$F_{\mathrm{blackbody}} = \sigma T^4$$

Planck Function:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/(\lambda kT)} - 1}$$

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(kT)} - 1}$$

Wien Peak Law for B_λ:

$$h\nu_{\mathrm{peak}} \approx 5kT$$

Wien Peak Law for B_ν:

$$h\nu_{\mathrm{peak}} \approx 3kT$$

Diffraction Limit:

$$\theta_{\mathrm{diff}} \approx \lambda/D$$

Energy Levels of Hydrogen:

$$E_n = -13.6\ \mathrm{eV}/n^2$$

Degeneracy of Neutral Hydrogen:

$$g_n = 2n^2$$

Maxwell-Boltzmann Distribution of Velocities:

$$dn/dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/(2kT)} 4\pi v^2$$

Ideal Gas Law:

$$P = nkT = \rho kT/(\mu m_{\mathrm{H}})$$

Relative Boltzmann Probabilities:

$$\frac{N_b}{N_a} = \frac{g_b}{g_a} e^{-(E_b - E_a)/kT}$$

Partition Function For Use in Saha Equation:

$$Z = \sum_i g_i e^{-(E_i - E_1)/(kT)}$$

Saha Equation (for state I vs. II; but generalizable):

$$\frac{n_{\mathrm{II}} n_e}{n_{\mathrm{I}}} = \frac{2 Z_{\mathrm{II}}}{Z_{\mathrm{I}}} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-\chi_1/kT}$$

Energy-Momentum for Photons:

$$E = cp$$

Optical Depth:

$$\tau = \int n \sigma dx$$

Hydrostatic Equilibrium:

$$\frac{dP}{dr} = -\rho g$$

1 Keck

The Keck telescope has a diameter D and operates at wavelengths λ. It observes a star of radius R_* a distance d away. Assume the star is a blackbody.

(a) [2 points] **Assuming the telescope is diffraction-limited, how close must the star be for its image to be resolved?** *A symbolic, order-of-magnitude answer for d suffices.*

Because the telescope is diffraction limited, its angular resolution ϕ is $\phi \approx \lambda/D$. To order of magnitude, we can resolve the star, when a single pixel covers the angular radius of the star (so that two pixels cover the whole star). The angular radius of the star is given by $\theta_* = R_*/d$. Equating θ_* with ϕ, yields a distance of $d \approx R_* D/\lambda$. This is the maximum distance at which the star is resolved; at closer distances, the star appears larger and is still resolved. So, $\boxed{d \lesssim R_* D/\lambda}$ to resolve the star.

(b) [2 points] **Assume the star is, in fact, resolved. At how many different wavelengths must the star's specific intensity be measured to infer the star's temperature T_*? Explain your reasoning BRIEFLY.**

The star is assumed to be a blackbody, so its specific intensity is given by the Planck function $B_\lambda(T_*)$. Thus, it is possible to infer the star's temperature T_* with $\boxed{\text{a measurement of } B_\lambda \text{ at a single wavelength}}$, by solving the equation for the Planck function for T_* using B_λ and λ.

(c) [2 points] Someone has the bright idea of using the Keck telescope as a night-time power generator by collecting starlight. **Assume that T_* is known. How much power P does the telescope collect from the star?** *Give an exact symbolic answer in terms of the variables given and fundamental constants.*

The luminosity of the star is the flux through the surface of the star times the surface area of the star (the so-called Stefan-Boltzmann law). This is equal to $L_* = \sigma T_*^4 4\pi R_*^2$. The star is a distance d from earth, so the flux measured on earth by the Keck telescope is

$$F_{\rm rec} = \frac{L_*}{4\pi d^2} = \sigma T_*^4 \frac{R_*^2}{d^2}.$$

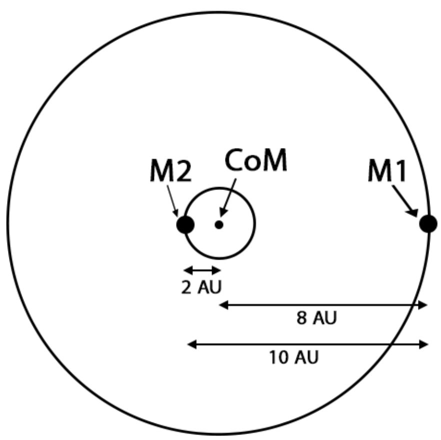
The power received P by the Keck telescope is a measure of the energy per unit time hitting the telescope's mirror. This can be found by multiplying the flux received by the area of the mirror, where $A_{\rm mirror} = \pi(D/2)^2$. So, the power collected from the star by the Keck telescope is

$$\boxed{P = \sigma T_*^4 \frac{\pi D^2 R_*^2}{4d^2}}.$$

2 Binary

Consider a binary star system of mass $M_1 = 1M_\odot$ *and* $M_2 = 4M_\odot$ *separated by a total distance of 10 AU. The binary orbit is circular.*

(a) [3 points] **Draw, to scale, the orbits of** M_1 **and** M_2 **about the common center-of-mass. Label all lengths in units of AU. Indicate the positions of** M_1 **and** M_2 **on your diagram.**



(b) [5 points] The center-of-mass (a.k.a. barycenter) moves away from the Earth at 10 km/s. Furthermore, the binary inclination is known to be $i = 60$ deg.

Calculate the maximum redshifted wavelength of the Hα line of M_1. Recall that the rest wavelength of Hα is 6563 Angstroms.

Redshift occurs due to the line-of-sight orbital speed of the M_1 as well as the speed of the barycentre. The projected orbital speed is

$$v_r = \frac{2\pi a_1 \sin i}{P} \tag{1}$$

where $a_1 = 8\text{AU}$ is the semi-major axis of M_1's orbit, and P is the orbital period which can be calculated from Kepler's 3rd law:

$$P = \left(\frac{4\pi^2 a^3}{G(M_1 + M_2)} \right)^{1/2} \tag{2}$$

where $a = a_1 + a_2 = 10\text{AU}$. Plugging in all the numbers, $v_r = 14.61\text{km s}^{-1}$. The difference between the rest-frame and the observed wavelength, by Doppler shift, is

$$\Delta\lambda = \frac{v_r + v_{bary}}{c} \lambda_{rest} \tag{3}$$

where $v_{bary} = 10\text{km s}^{-1}$, c is the speed of light, and $\lambda_{rest} = 6563\mathring{A}$, the rest-frame wavelength of Hα. Plugging in all the numbers, $\Delta\lambda = 0.54\mathring{A}$. The maximum redshifted wavelength of the Hα line of M_1 is then $\boxed{6563.54\mathring{A}}$. (You could have also used the mass function to get the line-of-sight orbital speed).

3 Hot Hydrogen

[6 points] Idealize hydrogen as having only 3 possible states: $n = 1$, $n = 2$, and ionized.

Much of the hydrogen in intergalactic space has temperature $T = 10^6$ K and a total number density $n_{\text{total}} = 10^{-3}$ cm^{-3}.

Most of the hydrogen is completely ionized. A tiny portion is neutral. **Of the NEUTRAL hydrogen, what percentage is in the $n = 2$ state?**

Hint: The energy difference between $n = 1$ and $n = 2$ is 10.2 eV.

The percentage of neutral hydrogen in the $n = 2$ state can be found by using the equation for the relative Boltzmann probabilities of two energy states

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} e^{-(E_2 - E_1)/kT},$$

where the subscripts 1 and 2 refer to the neutral hydrogen in the $n = 1$ and $n = 2$ states, respectively. We can solve the above equation knowing that the degeneracy for neutral hydrogen is given by $g_n = 2n^2$ and the energy difference between the $n = 2$ and $n = 1$ states is 10.2 eV. Plugging in the numbers yields $N_2/N_1 = 3.55$.

However, the problem is not asking for N_2/N_1, but N_2/N_{neutral}, where N_{neutral} is the total number of neutral hydrogen atoms. Using the fact that the neutral hydrogen can only be in the $n = 1$ or $n = 2$ states, we can write

$$\frac{N_2}{N_{\text{neutral}}} = \frac{N_2}{N_1 + N_2} = \frac{N_2/N_1}{1 + N_2/N_1}.$$

Plugging in the previous result for N_2/N_1, the percentage of neutral hydrogen in the $n = 2$ state is $\boxed{N_2/N_{\text{neutral}} = 78\%.}$

4 Journey to the Center of the Earth

The Earth has radius $R_\oplus$ and mass $M_\oplus$.

[3 points] (a) **Can you estimate the pressure P at the center of the Earth? If yes, give a symbolic expression in terms of the given variables and fundamental constants. If no, explain why. BE BRIEF.**

$\boxed{\text{YES, WE CAN}}$ (Mr. President). The Earth must support itself against its own gravity — otherwise the Earth would become a black hole. Pressure must be balancing gravity, according to hydrostatic equilibrium.

$$\frac{dP}{dr} = -\rho g \tag{4}$$

$$\frac{P}{R} \sim -\frac{M}{R^3}\frac{GM}{R^2} \tag{5}$$

$$P \sim \frac{GM^2}{R^4} \tag{6}$$

Finally plug in $M = M_\oplus$ and $R = R_\oplus$, so that $\boxed{P \sim GM_\oplus^2/R_\oplus^4}$

[3 points] (b) **Can you estimate the temperature T at the center of the Earth? If yes, give a symbolic expression in terms of the given variables and fundamental constants. If no, explain why. BE BRIEF.**

One is tempted to combine the above relation with the ideal gas law ($P = \rho k T/(\mu m_{\rm H})$) to estimate the temperature T. Such a procedure would yield $T \sim GM\mu m_{\rm H}/kR$.

One must resist this temptation, because the Earth is not an ideal gas.

Therefore, $\boxed{\text{NO}}$, we can't estimate the core temperature of the Earth $\boxed{\text{because the Earth is not an ideal gas.}}$ It is a rock.

5 Pressure at the Photosphere

[6 points] (a) Consider a medium that extends downward from $z = 0$ at its surface to $z = -\infty$ at depth. The medium has constant mass density ρ and mean molecular weight μ. Gravity points downward: $\vec{g} = -g\hat{z}$, where $g > 0$ is a constant. At $z = 0$, the pressure $P = 0$.

Each particle of the medium absorbs light with cross section σ.

An observer at $z > 0$ looks straight down into the medium. **Derive the pressure P where the observer's line of sight reaches an optical depth $\tau = 1$. Express using the variables given and fundamental constants.**[1]

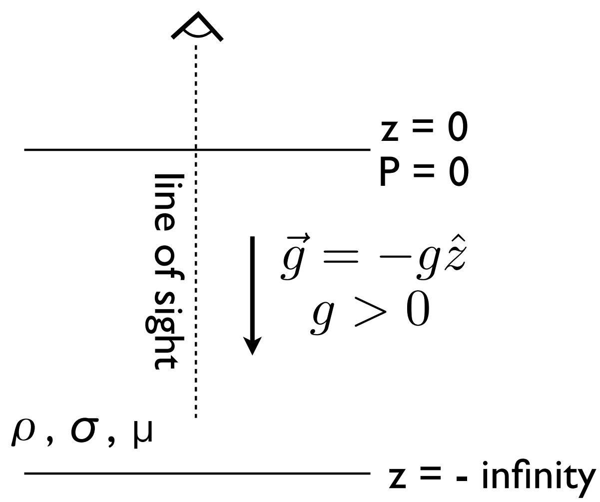


Figure 1: Observer looking down into a constant-density atmosphere.

[1] What you derive is called the "photospheric pressure." It is used routinely in stellar astrophysics as a boundary condition, and is valid to order-of-magnitude even when ρ and g are not constant.

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First let's use hydrostatic equilibrium to get the pressure P as a function of depth z. This is a problem straight out of the homework (see swimming pools).

$$\frac{dP}{dz} = -\rho g \tag{7}$$

$$dP = -\rho g\, dz \tag{8}$$

$$\int_0^P dP = -\int_0^z \rho g\, dz \tag{9}$$

$$P = -\rho g z \tag{10}$$

Notice $z < 0$ below the surface, so the pressure P is always positive, which makes sense.

Now what z corresponds to optical depth $\tau = 1$? By definition of τ (on the cover sheet),

$$\tau = -\int_z^0 n\sigma\, dz \tag{11}$$

Note how the integral runs from z (inside the atmosphere) to 0. That's because z increases upward ($dz > 0$) according to our sign convention.

We have:

$$\tau = \int_z^0 n\sigma\, dz \tag{12}$$

$$\tau = \int_z^0 n\mu m_{\mathrm{H}} \frac{\sigma}{\mu m_{\mathrm{H}}}\, dz \tag{13}$$

$$\tau = \int_z^0 \rho \frac{\sigma}{\mu m_{\mathrm{H}}}\, dz \tag{14}$$

$$\tau = \rho \frac{\sigma}{\mu m_{\mathrm{H}}} \int_z^0 dz \tag{15}$$

$$\tau = -\rho \frac{\sigma}{\mu m_{\mathrm{H}}} z \tag{16}$$

Set $\tau = 1$, and solve for $z = -\mu m_{\mathrm{H}}\sigma/\rho$. Finally plug this z into $P = -\rho g z$, and we get $\boxed{P = g\mu m_{\mathrm{H}}/\sigma}$.

For the experts, $\sigma/(\mu m_{\mathrm{H}}) \equiv \kappa$, the opacity (cross-section per unit mass). So the experts say: $P = g/\kappa$. This relation is used all the time in studies of stellar atmospheres.

6 BONUS (Optional): HOT CLUSTER GAS

[3 points] An X-ray telescope detects a giant blob of hydrogen plasma in a cluster of galaxies. The temperature of the plasma is $T = 10^6$ K, and the blob is roughly spherical with radius $R = 10$ Mpc (1 Mpc = 1 mega-parsec = 10^6 parsec).

Estimate to order-of-magnitude the total mass M contained in the cluster. Give your answer in solar masses ($M_\odot$). Explain your reasoning at every step.

Using virial theorem, we can find the mass using kinetic energy and the radius of the blob. Let's say the gas is made out of N hydrogen atoms. We know that the typical kinetic energy of EACH particle is ~kT. The total kinetic energy is then NkT. From the virial theorem,

$$\begin{aligned} < U > &= -2 < K > \\ -\frac{GM^2}{R} &\sim -2NkT \\ \frac{GM^2}{R} &\sim 2NkT \\ \frac{GM^2}{R} &\sim 2\frac{M}{m_H}kT \\ M &\sim \frac{2RkT}{Gm_H} \end{aligned} \tag{17}$$

where the second equation comes from $< K >= NkT$, fourth equation comes from $N = M/m_H$, and the fifth equation is simply rearrangement to isolate M to the left side of the equation. Plugging in the numbers, we get $\boxed{M = 3.7 \times 10^{13} M_\odot}$. You could have also used hydrostatic equilibrium and ideal gas law to get the same answer.

Our answer ignores some subtleties, one of which is actually quite important. One is that above derivation ignores electrons, which actually contribute just as much as kinetic energy as the protons (since EVERY particle has $\sim$kT). This is just a factor of 2 error, so it doesn't matter to order-of-magnitude. But the big thing our solution does not recognize (and which is impossible to infer from the limited information given in the problem) is that the mass we solved for is actually NOT hydrogen. It is actually DARK MATTER. The dark matter is confining just a little bit of hot hydrogen. We will see how this works in Astro 7B.