Astro 7A – Problem Set 10

1 Superstar

Stars whose masses approach $10^2 M_\odot$ are (1) supported by radiation pressure, and (2) entirely ionized so that photons Compton scatter off free electrons as they random walk from the core to the surface. The cross section for Compton scattering at the relevant photon energies is given approximately by the Thomson cross section.

(a) Using arguments similar to those presented in class, derive how radius R scales with mass M for such stars. A proportionality is sufficient.

(b) Derive how luminosity L scales with M for such stars.

2 I'm Your Venus

In class we stated that radiative diffusion is equivalent to the so-called "greenhouse effect" that heats the Earth's (and Venus's) atmosphere. This problem quantifies that statement.

Sunlight at visible wavelengths strikes the ground. The ground warms up and radiates in the infrared. The infrared radiation diffuses (random walks) back out into space. The optical thickness at infrared wavelengths is due to so-called greenhouse gases—water vapor, carbon dioxide, and methane—which are transparent in the visible but absorb strongly in the infrared.

Take the Sun to be directly overhead. Assume that the ground absorbs just 10% of the incident sunlight (because of reflective clouds and because the ground is also reflective), but that it re-radiates this energy as if it were a perfect blackbody. The radiation (both incoming and outgoing) is directed purely perpendicular to the ground, in a plane-parallel atmosphere.

(a) Calculate the temperature of the GROUND. Call this temperature $T_{\mathrm{eff,g}}$ and express in K. Just consider a small patch of the ground (don't worry about the entire surface of the Earth). Also don't forget that we are assuming that only 10% of the incident sunlight is absorbed.

(b) The actual temperature of AIR near the ground (NOT the ground, but the AIR RIGHT ABOVE the ground) CAN BE GREATER than $T_{\mathrm{eff,g}}$. The difference is due to the fact that the outgoing infrared radiation must travel through the air, which is optically thick at infrared wavelengths.

The equation of radiative diffusion[1] governs how the air temperature varies with height, under the assumption that the energy in the air is transported by photons:

$$\frac{dT}{dz} = -\frac{3}{4ac}\frac{\rho\kappa}{T^3}F_{\rm rad} \tag{1}$$

Decide what the outgoing flux $F_{\rm rad}$ is, as a function of height z above the ground. Express symbolically in terms of the variables given and fundamental constants.

In the context of this problem, $F_{\rm rad}(z)$ is the amount of OUTGOING energy crossing a unit area oriented parallel to the ground at a given height z, per unit time.

The atmosphere is in STEADY-STATE (radiative equilibrium; the atmosphere is not changing with time). Decide from this condition of balance how $F_{\rm rad}$ (the outgoing flux) varies with height. Explain carefully your reasoning.

Hint: The ground radiates infrared photons into the air. The air absorbs this infrared radiation, and radiates it into space. How much energy flux is the GROUND radiating into the air?

(c) Solve the equation of radiative diffusion, using the following definition of τ (measured from the ground up):

$$\tau(z) = \int_0^z \rho\kappa dz \tag{2}$$

Write down the solution for $T(\tau)$, using the boundary condition that for $\tau = \tau_{\rm max}$ ($z = \infty$), $T(\tau = \tau_{\rm max}) = T_{\rm eff,g}$.[2] Your solution for $T(\tau)$ should contain only $\tau_{\rm max}$, $T_{\rm eff,g}$, and τ.

(d) Given the actual air temperature near the ground of $T \approx 290\,{\rm K}$, calculate the (infrared) optical depth of the entire atmosphere, $\tau_{\rm max}$.

Human beings (in particular drivers of SUVs) are responsible for increasing the infrared optical depth $\tau_{\rm max}$ everyday, thereby raising the air temperature and wreaking havoc on global ecosystems.

(e) To order of magnitude, how many steps does an infrared photon take in random walking from the ground to space?

[1] See pages 353–354 of Ryden & Peterson, but keep in mind that their expressions pertain only to a spherical geometry, as in the interiors of stars, whereas this problem concerns a plane-parallel atmosphere. That is, for this problem, the ground is perfectly flat, and every layer of atmosphere above the ground is a perfectly flat plane that is parallel to the ground.

[2] The justification for this boundary condition is a little bit outside the scope of the class. In Astro 201, one learns that $T = T_{\rm eff}/2^{1/4}$ is more accurate. It may seem weird that the air infinitely far away from the ground has (nearly) the same temperature as the ground, but it is true.

3 The Adiabat

Here we derive the adiabatic temperature gradient. Remember that this adiabatic gradient DOES NOT EQUAL the actual gradient. The adiabatic gradient is a reference quantity, to be compared against the actual gradient to decide whether a gas is convective or not. The adiabatic gradient is derived under the assumption that each displaced fluid parcel (a.k.a. "blob") exchanges zero heat with its surroundings, contrary to what may happen in reality.

Use the adiabatic relation for (a dry, non-condensing) gas

$$P \propto \rho^{\gamma}\,, \tag{3}$$

the condition of hydrostatic equilibrium,

$$\frac{1}{\rho}\frac{dP}{dr} = -g \tag{4}$$

and the ideal gas law,

$$P = \frac{\rho k T}{\mu m_{\rm H}} \tag{5}$$

to solve for the adiabatic temperature gradient, $(dT/dr)_{\rm ad}$, in terms of g, k, μ, and γ.

Aside: This gradient is also called "the dry adiabatic lapse rate" in the atmospheric science literature—"dry" because we are not accounting for trace condensates, and the heat of condensation/evaporation associated with such trace species. The trace condensate could be water in the case of the Earth's atmosphere. Or it could be helium—which, like water, can form droplets if sufficiently cold—in the case of Jupiter's interior. If we did account for condensing trace species, the temperature gradient would instead be called the "wet adiabatic lapse rate."