Astro 7A – Problem Set 7

1 Time-Averaging the Potential Energy

The solution to this problem helped to motivate the virial theorem in class.

In general, an average of some continuous function $f(t)$ over an interval τ is given by

$$\langle f(t) \rangle = \frac{1}{\tau} \int_0^{\tau} f(t) dt \,. \tag{1}$$

Prove that, when averaged over one orbit,

$$\langle U \rangle = -G m_1 m_2 / a \tag{2}$$

where U is the gravitational potential energy of a binary system containing masses m_1 and m_2 in a relative orbit of semimajor axis a. You may find the following definite integral useful:

$$\int_0^{2\pi} \frac{d\theta}{1 + e\cos\theta} = \frac{2\pi}{\sqrt{1-e^2}} \tag{3}$$

2 The X-ray Binary SMC X-1

Embedded in the satellite galaxy called the Small Magellanic Cloud (SMC; visible to the naked eye on a dark night as a cloud-like patch of stars in the Southern Hemisphere) is one of the brightest X-ray sources in the night sky, SMC X-1.

SMC X-1 is an X-ray pulsar: an object that periodically beams X-rays toward the Earth. We say it "pulses" in X-rays, like a clock (or more accurately, a rotating lighthouse, as we learn later in the course). The average time between pulses is $P = 0.714890$ s.

However, the clock does not keep perfect time. The pulse period is observed to vary slightly, and sinusoidally with time: see smcx1.jpg on the ps7A.html website. Sometimes the observed period $P_{\rm obs}$ is faster than P, and sometimes slower. The period of the sine wave on the plot is $T = 3.89217$ days—not to be confused with P, the average pulse period! The plot gives the x-axis in units of orbital phase ranging from 0 to 1, where 1 corresponds to a total time of T.[1]

[1] The reason that orbital phase is plotted is because the data extend over many cycles (i.e., total elapsed time $t \gg T$). We say the data are "folded" on top of one another to make a single sine wave.

(a) Interpret the sinusoidal variation of the pulse period to be due to orbital motion of SMC X-1. Calculate a lower limit on the mass of the orbital companion of SMC X-1.

(b) Optical observations at the position of the X-ray source reveal a star (Sanduleak 160; we refer to it as the "optical companion"). Based on measurements of its spectral lines, the star's radial velocity varies as a near-perfect sine wave, with amplitude $K_C = 20$ km/s. From all the data presented so far (neglecting parts c and d below), what can we deduce about the individual masses of SMC X-1 and Sanduleak 160? Can we calculate the ratio of their masses? If so, what is this ratio?

(c) Suppose we know that the binary inclination $i = 90$ deg (the binary orbit is observed edge-on). What can we say now about the individual masses of SMC X-1 and Sanduleak 160?

(d) Notice on smcx1.jpg that there are no X-ray data in the region marked "Eclipse". That is because SMC X-1 is being eclipsed by Sanduleak 160 during those orbital phases. Still assuming that $i = 90$ deg,[2] and further assuming that SMC X-1 is much smaller in size compared to Sanduleak 160,[3] calculate the radius of Sanduleak 160, in solar radii.

3 Globular Cluster

A globular cluster is a collection of stars bound by gravity. Idealize the cluster as a sphere of radius $R = 1$ pc, inside of which stars are uniformly distributed.

By measuring how light from the cluster is Doppler shifted, it is determined that the typical speed of a star inside the cluster is $v = 20$ km/s.

Use the virial theorem to derive an expression for the total mass M of the cluster using the variables given and fundamental constants. Give both a symbolic expression AND a numerical evaluation in solar masses ($M_\odot$).

Precision in your symbolic answer will be rewarded (i.e., try to keep track of dimensionless constants on the order of unity).

[2] Actually i is closer to 60 deg, based on other observations and calculations.

[3] We will see this assumption is entirely justified, later in the course.