Astro 7A – Problem Set 9

1 Eureka, or "Crazy Naked Guy"

Consider a tank of water of total depth D, sitting on the Earth's surface. Take the surface of the water to be $z = 0$, and the bottom of the tank to be at $z = D$ (so z increases in the *downward* direction).

Assume the density of water is constant at ρ_{water} (we call the water "incompressible"). For low pressures, the assumption of incompressibility is excellent.

Take the pressure at the top of the water's surface to be zero (i.e., neglect atmospheric pressure).

(a) Calculate the water pressure as a function of depth, $P(z)$.

(b) A rectangular block of soap is placed in the water. The soap has a density $\rho_{\text{soap}} < \rho_{\text{water}}$. The soap is observed to float in water, in what appears to be a static configuration. The soap does not move, nor does the water.

Using the principles of hydrostatic balance (the same principles that are used to derive the equation of hydrostatic equilibrium), derive Archimedes' Principle, that the soap displaces a mass of water equal to its own mass.

Hint: Your answer in (a) holds regardless of whether the soap is present or not. That is, the lines of constant pressure in the tank are perfectly horizontal. If they were not, then there would a pressure difference horizontally across the tank, and horizontal currents would be driven. But that would violate the observation that the water in the tank is perfectly still—both vertically and horizontally.

(c) In geophysics, we may use the principle of hydrostatic equilibrium to explain differences in land elevations. In that context, the principle of hydrostatic equilibrium is called the principle of "isostatic equilibrium" (different word, same physics).

Place a block of wood next to the block of soap, and take $\rho_{\text{water}} > \rho_{\text{wood}} > \rho_{\text{soap}}$. The wood and soap have the same vertical thickness of h. Which rises higher, the soap or the wood, and by how much? Give a symbolic expression for the difference in heights, in terms of the variables given and fundamental constants.

In this way the geophysicists Pratt and Hayford surmised that different topographic heights corresponded to different underlying rock densities. The idea that the continents were essentially "blocks of soap and wood" (i.e., lithospheric or crustal blocks, a.k.a. tectonic plates) floating around in "water" (i.e., the mantle of the Earth) grew from this notion.

2 Black Hole Sun

(a) Suppose the temperature of all of the gas in the Sun were to cool instantly to zero degrees K. Estimate how long it would take for the Sun to collapse—to shrink its radius by a factor of 2. Express your answer both symbolically—in terms of $M_\odot$, $R_\odot$, and fundamental constants—and numerically, in seconds. An answer good to a factor of 3 suffices. Call this time the "free-fall time."

(b) Return to the normal hot Sun. Pressure disturbances in the Sun travel at the speed of sound, as given approximately by $v_{\rm sound} \approx \sqrt{kT/(\mu m_{\rm H})}$.[1]

Using the virial theorem, estimate the "sound-crossing time": the time for a sound wave to travel a solar radius. Express your approximate answer both symbolically—in terms of $M_\odot$, $R_\odot$, and fundamental constants—and numerically, in seconds. As with (a), your answer should only be approximate.

(c) Compare (a) to (b) and comment. If your answer in (a) were much shorter than your answer in (b), what would happen?

3 Optical Depth of Earth's Atmosphere

The sky is blue because sunlight at visible wavelengths scatters off nitrogen and oxygen molecules in the Earth's atmosphere. The scattering is called Rayleigh scattering. This problem computes the optical depth of the Earth's atmosphere to Rayleigh scattering.

Take the atmosphere to be composed of 80% N_2 and 20% O_2 by number. Assume the nitrogen and the oxygen are well-mixed throughout the atmosphere in these proportions.

Take the cross section to Rayleigh scattering of a single N_2 molecule to be $\sigma_{\rm N}$, and the cross section to Rayleigh scattering of a single O_2 molecule to be $\sigma_{\rm O}$.

The total pressure of the air at ground level is P_0. Take the atmosphere to be in hydrostatic equilibrium at constant temperature T, and constant gravity. The mass of the Earth is $M_\oplus$ and the radius of the Earth is $R_\oplus$.

(a) What is the mean molecular weight μ of the atmosphere?

(b) Derive a symbolic expression for the optical depth τ of the Earth's atmosphere to Rayleigh scattering. Measure the optical depth in the vertical direction: start from the ground and

[1] Note how the speed of sound happens to be comparable to the typical speeds with which individual particles move.

go straight up to infinity. Express your answer in terms of the variables given above and fundamental constants. This answer should be exact.

(c) Assume $\sigma_{\rm N} \approx 10^{-27}$ cm^2 and $\sigma_{\rm O} \approx 2 \times 10^{-27}$ cm^2. Looking up values for the other variables in your answer for (b), estimate the numerical value of τ. An answer good to within a factor of 2 will get full credit. Can an extraterrestrial observer far from the Earth see down to the ground at visible wavelengths?

4 Twinkle Twinkle Little T Dwarf

Brown dwarfs are "failed stars": objects insufficiently massive to fuse hydrogen via the pp chain. Yet they are still luminous. See the Wikipedia entry under "Brown dwarf." As shown in the first image on that webpage, Gliese 229B does not shine by reflected starlight, but instead shines from its own internal power source.

Consider a brown dwarf of mass $M = 20M_{\rm J}$ (20 $\times$ the mass of Jupiter), radius $R = 1R_{\rm J}$, luminosity $L = 10^{-5}L_\odot$, and age $t = 1$ Gyr.

(a) Can the brown dwarf derive its luminosity from chemical combustion, e.g., burning of carbon? As an extreme hypothesis, consider the object to be composed entirely of carbon, and say that each carbon atom can unleash 3 eV of energy. Give a quantitative reason why chemical reactions are a good or bad candidate for powering the dwarf.

(b) Can the brown dwarf derive its luminosity from gravitational contraction? Give a quantitative reason why or why not.