Astro 7B – Problem Set 2

1 Tidal Friction: Energy

We return to the same problem set-up as PS 1, Problem 1.

Two masses m_1 and m_2 orbit each other with semimajor axis a, with $m_2 \ll m_1$. The body m_1 has a rotational moment of inertia I_1 (about an axis that passes through its center of mass) and a spin angular frequency Ω_1. Treat m_2 as a point mass.

Consider the tide raised ON m_1 BY m_2 (e.g., the tide ON the Earth raised BY the Moon).

(a) Write down the total energy of the system, E, in terms of m_1, m_2, I_1, Ω_1, a, and fundamental constants. Your expression should have two terms. Work in the frame centered on m_1, as we did in PS 1, Problem 1.

(b) In PS 1, Problem 1b, you should have found:

$$\dot{a} = -\frac{2}{m_2}\left(\frac{a}{Gm_1}\right)^{1/2} I_1\dot{\Omega}_1 \tag{1}$$

as a consequence of the conservation of angular momentum ($\dot{L} = 0$). You are encouraged to review the solution set if this does not look familiar.

Although the total angular momentum does not change ($\dot{L} = 0$), the same is not true for $\dot{E}$ because there is friction. The time-dependent "cracking" of the surface of m_1 (if m_1 is rocky) — and sloshing of the oceans into fjords in the case of the Earth — causes $\dot{E} < 0$ (mechanical energy goes into heat, and ultimately radiation).

Evaluate $\dot{E}$ by taking the time derivative of (a), and combine with equation (1) above, to obtain

$$\dot{a} = \frac{2\dot{E}}{(\omega_2 - \Omega_1)}\frac{1}{m_2}\sqrt{\frac{a}{Gm_1}} \tag{2}$$

where ω_2 is the orbital angular frequency ($\omega_2 = 2\pi/P$, where P is the orbital period).

Thereby conclude that since $\dot{E} < 0$, the sign of $\dot{a}$ depends on the sign of $\Omega_1 - \omega_2$ (whether m_2 is at smaller semimajor axis or larger semimajor axis compared to the synchronous orbit with the spinning m_1). Fill in the BLANKS: If m_2 is outside synchronous orbit with the spinning m_1, then m_2's orbit BLANK. If m_2 is inside synchronous orbit with the spinning m_1, then m_2's orbit BLANK.

2 What's For Dinner?

In class we discussed how black holes above a certain mass can "swallow stars whole" without tidally disrupting them.

Estimate to order-of-magnitude the mass of the black hole M above which an adult human being can pass through the black hole's event horizon without being tidally disrupted.

Unlike stars, adult human beings (and other astrophysically small objects, like desks and trees and buildings and laptops, etc.) are not held together by gravity, but rather by electrostatic forces. Unfortunately, from this fact alone it is still too difficult to estimate from first principles the binding force holding together a human being. *In other words, it is not recommended to estimate this binding force by simply applying Coulomb's law.*[1] Instead, estimate the binding force using your own personal experience and/or experimentation. You can use the internet if you absolutely have to, but please first try doing this problem without consulting the web — it is more fun and educational that way. There are many ways to estimate the human binding force; you are encouraged to try a few and check that they agree to order-of-magnitude.[2]

3 Hill Sphere / Roche Lobe: A Mnemonic

Consider two masses m_1 and m_2 in a circular orbit with semimajor axis a, with $m_2 \ll m_1$. The "Hill sphere" or "Roche lobe" surrounding m_2 is a pear-shaped surface inside of which test particles can orbit m_2 (i.e., be "trapped" or bound to m_2). The characteristic radius of the Hill sphere is

$$a_{\rm Hill} = \left(\frac{m_2}{3m_1}\right)^{1/3} a\,. \tag{3}$$

Technically this is the distance from m_2 to the first Lagrange point L1, in the limit that $m_2 \ll m_1$.[3] We derived a rough order-of-magnitude version of this formula in class.

[1] The reason why a naive application of Coulomb's law fails is because ordinary objects (like human beings) are not perfect crystals with perfectly arranged positive and negative charges. Instead, ordinary objects are full of microscopic "flaws" that cause them to break far more easily than one might naively guess. Trying to model these flaws from first principles is hard (at least, too hard for your lecturer!)

[2] But don't try so many that you have to check yourself into a hospital!

[3] It is not a crime to call this the radius of the Roche lobe as well, although technically the radius of the Roche lobe is defined as the radius of the equivalent sphere containing a volume equal to that of the Roche lobe. Eggleton (1983) gives this radius as $a_{\rm Roche} = 0.49(m_2/m_1)^{1/3}a$ in the limit $m_2/m_1 \ll 1$, which you

Show that the orbital period P_2 of a test particle orbiting m_2 at a distance of $a_{\rm Hill}$ matches, to order-of-magnitude, the orbital period P_1 of a test particle orbiting m_1 at a distance of a. This quantifies what we mean when we say that the Hill sphere or Roche lobe boundary is where the influence of m_2 on a test particle is about as strong as the influence of m_1 on a test particle.

can see is approximately the same as $a_{\rm Hill}$. Eggleton also gives a general fitting formula valid for any value of m_2/m_1.