Astro 250 – Planetary Dynamics – Problem Set 3

Try at least 1 out of 3.

Readings: Murray & Dermott 6.1–6.4 (don't let the disturbing expansions disturb you), 6.7-6.9, 6.10.1

Problem 1. The Titan ringlet and the 1:0 Apsidal Resonance

The Colombo ringlet, also known informally as the Titan ringlet, is a narrow planetary ring around Saturn that sits within the 1:0 apsidal resonance established by the largest of the Saturnian moons, Titan. This means that the precession rate of the apsidal line of the ringlet matches the mean motion of Titan; Titan appears to pull the ring along.

Denote Titan's mass over Saturn's mass by $M_T/M = 2.366 \times 10^{-4}$, its semi-major axis by $a_T = 1.22 \times 10^6$ km, and its mean longitude by λ_T. Denote a single ring particle's semi-major axis by $a = 77871$ km, its eccentricity by $e = 2.6 \times 10^{-4}$, and its mean motion by $\Omega = 2.834 \times 10^{-4}$ rad/ s.

a) Write down, to leading order in e, the single term of the disturbing function due to Titan (the perturber) that represents the 1:0 apsidal resonance. Leave all variables in symbolic form (do not plug in numbers).

b) Use Lagrange's equations to compute $\dot{a}$, $\dot{e}$, and $\dot{\tilde{\omega}}$ for the ring particle. Express in terms of the constant $\eta = (M_T/M)\Omega\alpha H_{10}/2$, where $\alpha = a/a_T$ and $H_{10} = 2b_{1/2}^{(1)}(\alpha) + \alpha(d/d\alpha)b_{1/2}^{(1)}(\alpha) - 3\alpha$. We will use these expressions in a future problem set (#5) to compute the time evolution of e and $\tilde{\omega}$ explicitly.

Problem 2. Neptunian Resonances in the Kuiper Belt

The Kuiper Belt comprises a vast sea of planetesimals orbiting the Sun beyond Neptune ("trans-Neptunian space"). Many Kuiper Belt Objects (KBOs) have been discovered to inhabit Neptunian mean-motion resonances, most notably the outer 2:3 resonance and the outer 1:2 resonance. (In the literature, you will often see these referred to as the 3:2 and 2:1 resonances, respectively. Don't let it throw you.) Pluto inhabits the 2:3 resonance, along with thousands of other KBOs, collectively referred to as "Plutinos." Objects in the 1:2 resonance have been referred to as "Twotinos."

a) Write down, to first order in the eccentricity of the KBO, the resonant term of the disturbing function due to Neptune (the perturber) at its outer 1:2 resonance. Assume that all bodies lie in the invariable plane and that Neptune has zero eccentricity. Use Appendix B and follow the instructions on page 250 of Murray & Dermott to evaluate the indirect term.

b) Repeat for the 2:3 resonance. Numerically evaluate the ratio of (a) to (b), assuming the cosine factors both equal -1. What is mainly responsible for making the 2:3 resonance stronger than the 1:2? This result has consequences for the relative capture probabilities of KBOs into the two resonances as Neptune migrated its away into the ancient Belt (see Friedland 2001, ApJ, 547, 75; Chiang & Jordan 2002, submitted to AJ).

Problem 3. The Disturbing Function Referenced to Inertial Coordinates

This problem is derived from Goldreich & Tremaine's (1980, ApJ, 241, 425, hereafter GT) landmark treatise on disk-satellite interactions. This paper lays the foundation for understanding planetary ring shepherding and interactions between planets and circumstellar disks.

In a coordinate system that attaches the origin to the (primary) star of mass M, the perturbation potential due to a (secondary) planet of mass M_p reads

$$\phi^p(r,\theta,t) = -\frac{GM_p}{|\vec{r}-\vec{r}_p|} + \frac{GM_p}{|r_p|^3}\,\vec{r}_p \cdot \vec{r}$$

where $\vec{r}$ is the vector position (measured from the origin) where the potential is to be evaluated, and $\vec{r}_p$ is the vector displacement from the origin to the planet. Note that in equation (4) of GT, there is an error; their $(M_s/M_p)\Omega^2(r)$ should be replaced by GM_s/r_s^3. (This error is not propagated throughout the remainder of their paper.)

It is useful to expand ϕ^p in a Fourier series:

$$\phi^p(r,\theta,t) = \sum_{l=-\infty}^{\infty}\sum_{m=0}^{\infty} \phi^p_{l,m}(r)\cos\{m\theta - [m\Omega_p + (l-m)\kappa_p]t\}$$

where Ω_p is the mean angular frequency of the planet (the rotational frequency of the guiding center of the planet's orbit), and κ_p is the planet's epicyclic frequency (the frequency of radial oscillations due to non-zero eccentricity of the planet). In a frame that rotates at angular frequency $\Omega_p + (l-m)\kappa_p/m$, the perturbation potential is time-independent and has an m-fold azimuthal symmetry.

Assume that the planet's eccentricity is zero so that $|r_p|$ is a constant. Evaluate the strength of the "principal m^{th} component" of the potential, $\phi^p_{m,m}(r)$. This expression is sufficient to describe the perturbation potential of a planet on a perfectly circular orbit, and it is the component that establishes "principal Lindblad resonances" (Galacto-speak) or "first-order mean-motion resonances" (planeto-speak) in the disk. Principal Lindblad resonances excited in a disk dominate the evolution of the semi-major axis of the planet; they are responsible for planet migration. Express your answer in terms of Laplace coefficients (see the integral definition on page 237 of Murray and Dermott). Watch out for $m = 0$ and $m = 1$. Compare your answer to equation (7) of GT.