Astro 250: Solutions to Problem Set 3

by Eugene Chiang

Problem 1. *The Titan ringlet and the 1:0 Apsidal Resonance*

The Colombo ringlet, also known informally as the Titan ringlet, is a narrow planetary ring around Saturn that sits within the 1:0 apsidal resonance established by the largest of the Saturnian moons, Titan. This means that the precession rate of the apsidal line of the ringlet matches the mean motion of Titan; Titan appears to pull the ring along.

Denote Titan's mass over Saturn's mass by $M_T/M = 2.366 \times 10^{-4}$*, its semi-major axis by* $a_T = 1.22 \times 10^6\,\mathrm{km}$*, and its mean longitude by* λ_T*. Denote a single ring particle's semi-major axis by* $a = 77871\,\mathrm{km}$*, its eccentricity by* $e = 2.6 \times 10^{-4}$*, and its mean motion by* $\Omega = 2.834 \times 10^{-4}\,\mathrm{rad/s}$*.*

a) Write down, to leading order in e, the single term of the disturbing function due to Titan (the perturber) that represents the 1:0 apsidal resonance. Leave all variables in symbolic form (do not plug in numbers).

We are the ring; the perturber is outside. The disturbing function due to an exterior perturber is

$$R' = \frac{GM_T}{a_T}(R_D + \alpha R_E) \tag{1}$$

where $\alpha = a/a_T$, R_D is the direct contribution to the disturbing function, and R_E is the indirect contribution. From the situation at hand, we know the dominant cosine term must be

$$\cos(\tilde{\omega} - \lambda_T) \tag{2}$$

To get the strength of the direct term, look at Murray & Dermott's Table B.4; the relevant term is 4D1.1 where $j = 1$. The leading term is ef_{27}, where f_{27} is given in Table B.7 and equation (B.1). The strength of the indirect term is given by Table B.5; the relevant term is 4E1.2. The leading indirect term is $3e/2$. Putting it all together, we write

$$\begin{aligned} R' &= \frac{GM_T}{a_T}[ef_{27}\cos(\tilde{\omega} - \lambda_T) + \alpha\frac{3}{2}e\cos(\tilde{\omega} - \lambda_T)] & (3) \\ &= \frac{GM_T}{a_T}e\cos(\tilde{\omega} - \lambda_T)\left\{-\frac{1}{2}\left[2 + \alpha\frac{d}{d\alpha}\right]b_{1/2}^{(1)}(\alpha) + \frac{3}{2}\alpha\right\} & (4) \end{aligned}$$

$$= -\frac{GM_T}{2a_T} e\cos(\tilde{\omega} - \lambda_T)[2b_{1/2}^{(1)}(\alpha) + \alpha\frac{d}{d\alpha}b_{1/2}^{(1)}(\alpha) - 3\alpha] \quad (5)$$

$$= \boxed{-\tfrac{GM_T}{2a_T} e\cos(\tilde{\omega} - \lambda_T)H_{10}(\alpha)} \quad (6)$$

where we have looked ahead to part (b) of this problem for the definition of H_{10}.

b) Use Lagrange's equations to compute $\dot{a}$, $\dot{e}$, and $\dot{\tilde{\omega}}$ for the ring particle. Express in terms of the constant $\eta = (M_T/M)\Omega\alpha H_{10}/2$, where $\alpha = a/a_T$ and $H_{10} = 2b_{1/2}^{(1)}(\alpha) + \alpha(d/d\alpha)b_{1/2}^{(1)}(\alpha) - 3\alpha$. We will use these expressions in a future problem set (#5) to compute the time evolution of e and $\tilde{\omega}$ explicitly.

Lagrange's equations say $\dot{a} \propto \partial R'/\partial\lambda$. Since R' has no dependence on the mean longitude of the ring particle, $\boxed{\dot{a} = 0}$. To leading order, Lagrange's equation for $\dot{e}$ is

$$\dot{e} = \frac{-1}{\Omega a^2 e}\frac{\partial R'}{\partial\tilde{\omega}} \quad (7)$$

$$= -\frac{1}{\Omega a^2}\frac{GM_T}{2a_T}H_{10}(\alpha)\sin(\tilde{\omega} - \lambda_T) \quad (8)$$

Apply our usual trick of multiplying numerator and denominator by Ω to find

$$\boxed{\dot{e} = -\tfrac{\Omega M_T H_{10}(\alpha)a}{2Ma_T}\sin(\tilde{\omega} - \lambda_T) = -\eta\sin(\tilde{\omega} - \lambda_T)} \quad (9)$$

Finally, repeat for $\dot{\tilde{\omega}}$:

$$\dot{\tilde{\omega}} = \frac{1}{\Omega a^2 e}\frac{\partial R'}{\partial e} \quad (10)$$

$$\boxed{\dot{\tilde{\omega}} = -\tfrac{\eta}{e}\cos(\tilde{\omega} - \lambda_T)} \quad (11)$$

Problem 2. *Neptunian Resonances in the Kuiper Belt*

The Kuiper Belt comprises a vast sea of planetesimals orbiting the Sun beyond Neptune ("trans-Neptunian space"). Many Kuiper Belt Objects (KBOs) have been discovered to inhabit Neptunian mean-motion resonances, most notably the outer 2:3 resonance and the outer 1:2 resonance. (In the literature, you will often see these referred to as the 3:2 and 2:1 resonances, respectively. Don't let it throw you.) Pluto inhabits the 2:3 resonance, along with thousands of other KBOs, collectively referred to as "Plutinos." Objects in the 1:2 resonance have been referred to as "Twotinos."

a) Write down, to first order in the eccentricity of the KBO, the resonant term of the disturbing function due to Neptune (the perturber) at its outer 1:2 resonance. Assume that all bodies lie in the invariable plane and that Neptune has zero eccentricity. Use Appendix B and follow the instructions on page 250 of Murray & Dermott to evaluate the indirect term.

We are the Kuiper Belt Object; the perturber, Neptune, lies interior to us. Therefore the disturbing function takes the form

$$R = \frac{Gm_N}{a_N}(\alpha R_D + \frac{1}{\alpha}R_I) \tag{12}$$

where R_D is the direct term, R_I is the indirect term, and $\alpha = a_N/a$, the ratio of semi-major axes of Neptune and the KBO.

The relevant cosine term for the 1:2 resonance is $\cos(2\lambda' - \lambda - \tilde{\omega}')$, where primes denote the exterior Kuiper Belt Object (KBO) and unprimes denote the interior Neptune. From Table B.4, term 4D1.2, $j = 2$, and Table B.7, the strength of the direct term is $e' f_{31} = e'\frac{1}{2}[-1 + 4 + \alpha(d/d\alpha)]b^{(1)}_{1/2}(\alpha)$. From Table B.6, term 4I1.3, the strength of the indirect term is, to leading order, $-e'/2$. Putting it all together,

$$\boxed{R_{1:2} = \tfrac{Gm_N}{a_N}\cos(2\lambda' - \lambda - \tilde{\omega}')\tfrac{e'}{2}\left\{3\alpha b^{(1)}_{1/2}(\alpha) + \alpha^2(d/d\alpha)b^{(1)}_{1/2}(\alpha) - \alpha^{-1}\right\}} \tag{13}$$

b) Repeat for the 2:3 resonance. Numerically evaluate the ratio of (a) to (b), assuming the cosine factors both equal -1. What is mainly responsible for making the 2:3 resonance stronger than the 1:2? This result has consequences for the relative capture probabilities of KBOs into the two resonances as Neptune migrated its away into the ancient Belt (see Friedland 2001, ApJ, 547, 75; Chiang & Jordan 2002, AJ, in press).

The procedure is the same as for (a) except that now we plug in $j = 3$. The direct strength is $e' f_{31} = (e'/2)(-1 + 6 + \alpha(d/d\alpha))b^{(2)}_{1/2}(\alpha)$. And for the first-order eccentricity-type 2:3 resonance, there is no indirect term! Therefore

$$\boxed{R_{2:3} = \tfrac{Gm_N}{a_N}\cos(3\lambda' - 2\lambda - \tilde{\omega}')\tfrac{e'}{2}\left\{5\alpha b^{(2)}_{1/2}(\alpha) + \alpha^2(d/d\alpha)b^{(2)}_{1/2}(\alpha)\right\}} \tag{14}$$

Now we proceed to the numerical evaluation of the ratio, $R_{1:2}/R_{2:3}$. If we assume each cosine term equals -1 (this is the condition of exact resonance; the resonant argument $\phi = \pi$; the metronome is pointing up), then all the factors cancel except for those in the curly brackets. Now we need to evaluate some Laplace coefficients and their derivatives. Use equation (6.70) to see that

$$\frac{d}{d\alpha}b_{1/2}^{(1)} = \frac{1}{2}(b_{3/2}^{(0)} - 2\alpha b_{3/2}^{(1)} + b_{3/2}^{(2)}) \tag{15}$$

$$\frac{d}{d\alpha}b_{1/2}^{(2)} = \frac{1}{2}(b_{3/2}^{(1)} - 2\alpha b_{3/2}^{(2)} + b_{3/2}^{(3)}) \tag{16}$$

With these relations we can convert the derivatives in the curly brackets into pure Laplace coefficients. Now $\alpha_{1:2} = (1/2)^{2/3} = 0.6300$. And $\alpha_{2:3} = (2/3)^{2/3} = 0.7631$. Use equation (6.67) to evaluate a whole slew of Laplace coefficients. I wrote an IDL routine to do this; I used the numerical integrator, "qromb" (Romberg integration). Anybody who would like a copy of the IDL routine should feel free to contact me. One point to note: using the series solution (6.68) is not a good way to calculate these Laplace coefficients, because α is not much less than 1 for either the 1:2 or the 2:3 resonance. One would need to keep something like 10 terms in the series to get reasonable accuracy.

Anyway, it wasn't too painful to calculate that $\boxed{R_{1:2}/R_{2:3} = 0.5402/3.720 = 0.15}$. This number is as low as it is because of the indirect term's contribution to $R_{1:2}$; if the indirect term were omitted, the ratio would be $2.13/3.720 = 0.6$. Notice the indirect term acts to weaken the strength of the 1:2 resonance; it comes in with a minus sign in equation (13). Friedland (2001) has claimed that it is because of the indirect term's weakening of the 1:2 resonance that this resonance captures KBOs less efficiently than the 2:3 resonance for a given migration timescale.

Problem 3. *The Disturbing Function Referenced to Inertial Coordinates*

This problem is derived from Goldreich & Tremaine's (1980, ApJ, 241, 425, hereafter GT) landmark treatise on disk-satellite interactions. This paper lays the foundation for understanding planetary ring shepherding and interactions between planets and circumstellar disks.

In a coordinate system that attaches the origin to the (primary) star of mass M, the perturbation potential due to a (secondary) planet of mass M_p reads

$$\phi^p(r,\theta,t) = -\frac{GM_p}{|\vec{r}-\vec{r}_p|} + \frac{GM_p}{|r_p|^3}\,\vec{r}_p\cdot\vec{r}$$

where $\vec{r}$ is the vector position (measured from the origin) where the potential is to be evaluated, and $\vec{r}_p$ is the vector displacement from the origin to the planet. Note that in equation (4) of GT, there is an error; their $(M_s/M_p)\Omega^2(r)$ should be replaced by GM_s/r_s^3. (This error is not propagated throughout the remainder of their paper.)

It is useful to expand ϕ^p in a Fourier series:

$$\phi^p(r,\theta,t) = \sum_{l=-\infty}^{\infty}\sum_{m=0}^{\infty}\phi_{l,m}^p(r)\cos\{m\theta - [m\Omega_p + (l-m)\kappa_p]t\}$$

where Ω_p is the mean angular frequency of the planet (the rotational frequency of the guiding center of the planet's orbit), and κ_p is the planet's epicyclic frequency (the frequency of radial oscillations due to non-zero eccentricity of the planet). In a frame that rotates at angular frequency $\Omega_p + (l-m)\kappa_p/m$, the perturbation potential is time-independent and has an m-fold azimuthal symmetry.

Assume that the planet's eccentricity is zero so that $|r_p|$ is a constant. Evaluate the strength of the "principal m^{th} component" of the potential, $\phi^p_{m,m}(r)$. This expression is sufficient to describe the perturbation potential of a planet on a perfectly circular orbit, and it is the component that establishes "principal Lindblad resonances" (Galacto-speak) or "first-order mean-motion resonances" (planeto-speak) in the disk. Principal Lindblad resonances excited in a disk dominate the evolution of the semi-major axis of the planet; they are responsible for planet migration. Express your answer in terms of Laplace coefficients (see the integral definition on page 237 of Murray and Dermott). Watch out for $m = 0$ and $m = 1$. Compare your answer to equation (7) of GT.

If the planet has no eccentricity, the problem is relatively easy: in the frame rotating with the planet, the planet is fixed. Let's go into this rotating frame, and fix the planet to be on the x-axis in this frame. Define a to be the semi-major axis of the planet, so that the planet is located at $x = a$. Take ψ to be the angle between the x-axis and the position vector, $\vec{r}$, where we wish to evaluate the potential. By the law of cosines, $|\vec{r}-\vec{r}_p|^2 = r^2 + a^2 - 2ar\cos\psi$. And $\vec{r}_p\cdot\vec{r} = ar\cos\psi$. Then we can re-write the potential in terms of coordinates in this rotating frame as

$$\phi^p(r,\theta,t) = -\frac{GM_p}{|\vec{r}-\vec{r}_p|} + \frac{GM_p}{|r_p|^3}\vec{r}_p\cdot\vec{r} \tag{17}$$

$$= -\frac{GM_p}{(r^2+a^2-2ar\cos\psi)^{1/2}} + \frac{GM_p}{a^3}ar\cos\psi \tag{18}$$

Now let's examine the Fourier series. The problem asks us to examine $l = m$. Then

$$\phi^p(r,\theta,t) = \sum_{m=0}^{\infty}\phi^p_{m,m}(r)\cos\{m[\theta-\Omega_p t]\} \tag{19}$$

Recognize that $\psi = \theta - \Omega_p t$ (θ is the position angle in inertial space.) To get the Fourier amplitude, $\phi^p_{m,m}(r)$, multiply the above by $\cos m\psi$ and integrate over ψ from 0 to 2π:

$$\int_0^{2\pi}\phi^p(r,\psi)\cos m\psi\, d\psi = \int_0^{2\pi}\phi^p_{m,m}(r)\cos^2 m\psi\, d\psi = \pi\phi^p_{m,m}\ (\text{if m} \neq 0) \tag{20}$$

$$= 2\pi\phi^p_{m,m}\ (\text{if m} = 0) \tag{21}$$

where we have recalled the wonderful orthogonality property that $\int_0^{2\pi} \cos m\psi \cos n\psi \, d\psi = 0$ if $m \neq n$. Apply the same operation to (18): multiply (18) by $\cos m\psi$ and integrate over a full cycle of ψ:

$$\begin{aligned} (1+\delta_{m,0})\pi\phi^p_{m,m} &= -GM_p \int_0^{2\pi} \frac{\cos m\psi}{(r^2+a^2-2ar\cos\psi)^{1/2}}\, d\psi \\ &\quad + \frac{GM_p ar}{a^3} \int_0^{2\pi} \cos\psi \cos m\psi \, d\psi \qquad (22) \\ &= -\frac{GM_p}{a} \int_0^{2\pi} \frac{\cos m\psi}{(\beta^2+1-2\beta\cos\psi)^{1/2}}\, d\psi \\ &\quad + \frac{GM_p r}{a^2} \int_0^{2\pi} \cos\psi \cos m\psi \, d\psi \qquad (23) \end{aligned}$$

where $\beta = r/a$. If $m \neq 1$, then the second integral vanishes. For the first integral, we adopt the physicist's convention that for any integral we can't do, we bequeath it a special name. Let's call it a "Laplace coefficient."

$$b^{(m)}_{1/2} \equiv \frac{1}{\pi} \int_0^{2\pi} \frac{\cos m\psi \, d\psi}{(1-2\beta\cos\psi+\beta^2)^{1/2}} \qquad (24)$$

If $m = 1$, then the second integral in (23) equals

$$\begin{aligned} \frac{GM_p r}{a^2} \int_0^{2\pi} \cos\psi \cos m\psi \, d\psi &= \frac{GM_p r}{a^2}\pi \qquad (25) \\ &= \frac{GM_p}{a}\pi\beta \qquad (26) \end{aligned}$$

Putting it all together,

$$\boxed{\phi^p_{m,m} = -\tfrac{GM_p}{a}\left\{ b^{(m)}_{1/2}(1+\delta_{m,0})^{-1} - \delta_{m,1}\beta \right\}} \qquad (27)$$

Congratulations: you have derived equation (7) of Goldreich & Tremaine (1980) (Only 103 equations left to go). GT go on to derive expressions for $\phi^p_{m\pm1,m}$, the strengths of leading-order perturbations due to the finite eccentricity of the perturber; these will give rise to so-called "first-order Lindblad resonances" and "first-order co-rotation resonances;" here "first-order" is first-order in the eccentricity of the perturber.

Notice that in $\phi^p_{m,m}$, the indirect term only kicks in for $m = 1$, and it serves to weaken the potential, as we saw numerically in problem 2 of this set.