Astro 250 – Planetary Dynamics – Problem Set 4

Try at least 1 out of 3.

Readings: Murray & Dermott 7.1–7.2, 7.4–7.5, 7.7–7.8, 7.10–7.11

Problem 1. Precessing Planes and the Invariable Plane

Consider a star of mass m_c orbited by two planets on nearly circular orbits. The mass and semi-major axis of the inner planet are m_1 and a_1, respectively, and those of the outer planet, $m_2 = (1/2) \times m_1$ and $a_2 = 4 \times a_1$. The mutual inclination between the two orbits is $i \ll 1$. This problem explores the inclination and nodal behavior of the two planets.

a) What is the inclination of each planet with respect to the invariable plane of the system? The invariable plane is perpendicular to the total (vector) angular momentum of all planetary orbits. Neglect the contribution of orbital eccentricity to the angular momentum. Call these inclinations i_1 and i_2.

b) Take for the rest of this problem your reference plane to be the invariable plane. Use lowest-order secular theory to compute the frequencies of nodal precession of each of the two planets, $\dot{\Omega}_1$ and $\dot{\Omega}_2$. Check that these rates do not have any dependence on i_1 and i_2. They should depend only on the masses and semi-major axes. Be sure to include the sign.

c) How do i_1, i_2, and i vary in time?

d) Place a test particle in the invariable plane on a circular orbit at a semi-major axis of $a_t = 2 \times a_2$. Does the inclination of the particle with respect to the invariable plane remain zero? If you laid a disk of test particles in the invariable plane, would it remain there?

Problem 2. The Warp Radius of the Laplacian Plane

By now we are used to the idea that an oblate planet induces apsidal (and nodal) precession in a satellite's orbit. This problem explores the effect of the parent star on satellite precession rates.

Consider a star-planet-satellite system. Recognize that from the satellite's point-of-view, the parent star appears to revolve around the planet; the star can be considered merely another (very massive) satellite on an exterior orbit about the planet. (Who says that the Sun doesn't revolve around the Earth?)

a) Use secular theory to derive the rate of nodal precession of the satellite induced by the star, $\dot{\Omega}_c$. Take the mass of the star to be m_c, the planet mass to be m_p, the satellite mass to be zero, the satellite's distance to the planet to be a, and the planet's distance to the star to be r. Assume that the satellite's orbit plane about the planet is at low, but non-zero, inclination with respect to the planet's orbit plane about the star.

b) Take the leading term of equation (6.250) of Murray & Dermott for the nodal precession rate of the satellite induced by the planet's J_2 (oblateness). Those of you who solved problem 1 of problem set #1 should find the leading term of this expression not surprising. Compare this rate, $\dot{\Omega}_p$, to $\dot{\Omega}_c$ and solve for the critical planetocentric radius, a_w, at which these rates are equal. This is approximately the radius where the Laplacian plane warps. The Laplacian plane is that plane about which test particles nodally precess. At $a < a_w$, the Laplacian plane aligns itself with the planet's equator plane. At $a > a_w$, the Laplacian plane aligns itself with the planet's orbital plane.

c) Is the Earth's Moon inside or outside a_w? Repeat for the Saturnian satellites, Mimas and Titan.

Problem 3. Ring locking

Narrow rings encircle Uranus and Saturn that are apsidally locked. That is, for a given ring, the inner elliptical edge of the ring is observed to be nearly perfectly apsidally aligned with the outer elliptical edge of the ring. Apsidal locking is puzzling because planetary oblateness (J_2) induces differential apsidal precession across the ring. The inner edge wants to precess faster than the outer edge (because the former sits at a smaller semi-major axis than the latter); the edges would precess into one another on fast ($\sim 10^2$ yr) timescales; streamlines would cross, and the eccentricities of ring particles would be collisionally damped to zero. But the eccentricities of rings are not zero. How can a given ring maintain apsidal lock and precess about the planet as if it were a rigid body? Those of you who witnessed the Hubble Space Telescope movie of the Uranian epsilon ring in the first class know first-hand that indeed that ring is eccentric and that it rigidly precesses, giving rise to the "pulsing" effect in the movie. This problem takes a first qualitative step towards understanding apsidal locking. [Those of you who are really interested can read Chiang & Goldreich (2000, ApJ) or chapter 7 of my thesis. This problem was a real bear. But it is the happiest research problem I have worked on so far, and despite much sweat and toil over 2 decades by great dynamicists it remains incompletely solved.]

Idealize a given elliptical ring by two infinitesimally narrow, massive elliptical wires that are in the same plane and that are perfectly apsidally aligned. Take the eccentricity and semi-major axis of the inner wire to be e_1 and a_1, respectively, and those of the outer wire to be e_2 and a_2. Take e_1, $e_2 \ll 1$ and $\Delta a \equiv a_2 - a_1 \ll a_1, a_2$.

Put both massive wires around an oblate planet. The mutual gravitational attraction of the wires induces differential precession. Planetary oblateness induces differential precession. In fact, the two effects can exactly balance. Use the form of Gauss's equation and your knowledge of secular theory to deduce the sign of $\Delta e \equiv e_2 - e_1$. In other words, for the wire ringlets to remain apsidally aligned, must the distance between the two ringlets be smallest at apoapse or smallest at periapse?