

Try at least 1 out of 3.

Readings: Murray & Dermott Chapter 8 EXCEPT for 8.8–8.10, 8.15–8.20. Skim 8.11–8.14. And, of course, the journal articles that are on the web.

Problem 1. The Titan ringlet: Part II

This problem follows on the heels of problem 1 in Problem Set 3.

The Colombo ringlet, also known informally as the Titan ringlet, is a narrow planetary ring around Saturn that sits near the 1:0 apsidal resonance established by the largest of the Saturnian moons, Titan. This means that the precession rate of the apsidal line of the ringlet is very close to the mean motion of Titan.

Denote Titan’s mass over Saturn’s mass by $M_T/M = 2.366 \times 10^{-4}$, its semi-major axis by $a_T = 1.22 \times 10^6$ km, and its mean longitude by λ_T . Denote a single ring particle’s semi-major axis by $a = 77871$ km, its eccentricity by $e = 2.6 \times 10^{-4}$, and its mean motion by $\Omega = 2.834 \times 10^{-4}$ rad/s.

In problem 1 of PS 3, one showed that for the ring particle near the resonance,

$$\dot{a} = 0 \tag{1}$$

$$\dot{e} = -\eta \sin \phi \tag{2}$$

$$\dot{\tilde{\omega}} = -\frac{\eta}{e} \cos \phi \tag{3}$$

$$\phi \equiv \tilde{\omega} - \lambda_T \tag{4}$$

where $\eta = (M_T/M)\Omega\alpha H_{10}/2$, $H_{10} = 2b_{1/2}^{(1)}(\alpha) + \alpha(d/d\alpha)b_{1/2}^{(1)}(\alpha) - 3\alpha$, and $\alpha = a/a_T$ are constants.

The above expression for $\dot{\tilde{\omega}}$ is incomplete because it only accounts for mean-motion resonant forcing by Titan. What is missing is forcing by the local secular potential. Let’s just say:

$$\dot{\tilde{\omega}} = -\frac{\eta}{e} \cos \phi + \dot{\tilde{\omega}}_{sec} \tag{5}$$

where $\dot{\tilde{\omega}}_{sec} = \dot{\tilde{\omega}}_{\text{Saturn}} + \dot{\tilde{\omega}}_{\text{stuff}}$ is the total *additional* precession rate induced by the oblateness of Saturn and the secular potential of everything else—nearby rings (remember the

Titan ringlet is just 1 narrow ring embedded in Saturn's gigantic ring complex), Titan, other satellites, the Sun, lost pens, etc. In fact, the first of these contributions completely overwhelms the other contributions. There is no need to write out explicitly what all these terms are; we will just work with $\dot{\tilde{\omega}}_{sec}(a)$. (Those of you who did problem 1 of PS 1 know what this function is, but the explicit form is not needed for this problem!)

Further define

$$\epsilon(a) \equiv \dot{\tilde{\omega}}_{sec}(a) - \Omega_T \quad (6)$$

and a special semi-major axis, $a = a_0$, such that $\epsilon(a_0) = 0$.

a) Based on the above, your notes, and/or problem 1 of PS 1, do you need to modify the equation for \dot{e} to account for the secular potential?

b) Equations (2) and (5) are coupled ordinary differential equations. Replace e and ϕ in favor of the variables,

$$h \equiv e \cos \phi \quad (7)$$

$$k \equiv e \sin \phi \quad (8)$$

Write down \dot{h} and \dot{k} in terms of η , ϵ , h , and k .

c) Solve your equations for \dot{h} and \dot{k} . Your solution should contain two arbitrary constants: an amplitude and a phase associated with a sinusoidal oscillation.

d) Plot possible trajectories in h and k space. Identify the conditions under which ϕ is circulating or librating. If the particle is librating, what are the libration centers, $\langle \phi \rangle$? What libration centers are associated with $a > a_0$? What centers are associated with $a < a_0$? If you have the correct solution, you should notice that something terrible happens at $a = a_0$. This is simply a deficiency of our low-order theory.

Problem 2. Pluto and Neptune

Pluto inhabits two resonances: the 3:2 resonance for which $\phi = 3\lambda_P - 2\lambda_N - \tilde{\omega}_P$ librates about π with an amplitude ($\max(\phi) - \pi$) of 76° , and the Kozai resonance for which ω (the argument of perihelion) librates about $\pi/2$ with an amplitude ($\max(\omega) - \pi/2$) of 23° . The inclination of Pluto's orbit to Neptune's is about 16° . The semi-major axis of Neptune is 30.1 AU and the semi-major axis of Pluto is 39.4 AU.

a) Pluto's eccentricity is 0.25. If this eccentricity grew from a primordial value of (say) 0.01 via resonant capture by a migrating Neptune and continued migration, what

is the minimum distance over which Neptune must have migrated in semi-major axis? Give your answer in AU. Does your answer change substantially if Pluto's eccentricity were initially 0.05? 0.001?

b) Estimate a lower limit to the timescale, a_N/\dot{a}_N , over which this migration occurred.

c) What is the closest distance of approach that Pluto makes to Neptune? Recognize that the libration period of the 3:2 resonance is much shorter than the libration period within the Kozai resonance, so that you don't need to worry about commensurabilities. Compare your answer to the Hill sphere radius of Neptune.

Problem 3. N petals, forced eccentricities, and another definition of a resonant width

This problem generalizes problem 1 of this set. It is relevant for the resonant edges of planetary rings.

The edges of planetary rings are near principal Lindblad resonances of azimuthal wavenumber m established by shepherd satellites. At the exact resonance location,

$$(m \mp 1)n - mn_p \pm \dot{\omega} = 0. \quad (9)$$

Here m is a positive integer, n and n_p are the mean motions of a ring (test) particle and of the perturbing shepherd, and $\dot{\omega}$ is the apsidal precession rate of the ring particle. The upper/lower signs correspond to inner/outer Lindblad resonances.

Take the shepherd to be outside the ring. The resonant disturbing function of the shepherd is

$$R_{p,res} = \frac{Gm_p}{a_p} f(\alpha) e \cos \phi \quad (10)$$

$$\phi = (m - 1)\lambda - m\lambda_p + \tilde{\omega} \quad (11)$$

where λ 's are mean longitudes, e is the eccentricity of the test particle, and $f(\alpha) = f(a/a_p)$ is a dimensionless function of the ratio of semi-major axes of the particle to the perturber. f is often of order unity.

a) Calculate $\dot{\omega}_{res}$ and \dot{e}_{res} from $R_{p,res}$ using Lagrange's planetary equations. (We are neglecting the variation in semi-major axis because we emphasized in lecture that the fractional variation in a is e^2 smaller than the fractional variation in e .)

b) It is evident that $\dot{\phi} = (m - 1)n - mn_p + \dot{\omega}$. In reality, $\dot{\omega} = \dot{\omega}_{res} + \dot{\omega}_{sec}$. For this problem, we will consider $m \neq 1$ and say that $\dot{\omega}_{sec} \ll \dot{\omega}_{res}$. Note that we cannot ignore

$\dot{\omega}_{sec}$ if $m = 1$; see problem 1 above. Many planetary rings have their edges located at $m \sim 10$.

Similarly ignore \dot{e}_{sec} .

Define $\epsilon(a) = (m - 1)n - mn_p$ to write

$$\dot{\phi} = \epsilon(a) + \dot{\omega}_{res} \quad (12)$$

Now take the particle to be firmly in the resonance with vanishingly small libration amplitude; that is, consider the limit $\dot{e} \rightarrow 0$ and $\dot{\phi} \rightarrow 0$. What are the equilibrium values for e and ϕ ? The value for e that you have deduced is called the “forced eccentricity” (as opposed to the “free eccentricity,” which is the amplitude of libration in h - k space; see problem 1 above). Remember that $\epsilon(a)$ can be either negative or positive, so you should never get a negative eccentricity.

c) Express the eccentricity e in terms of the distance, $x = a - a_0$, where $(m - 1)n(a_0) = mn_p$. Of course, we are considering $x \ll a_0$.

d) In the frame co-rotating with the shepherd (which we take to be moving on a perfectly circular orbit), SKETCH APPROXIMATELY the trajectories of ring particles for a few values of x , both positive and negative. You may find it helpful to think in terms of epicyclic frequency, $\kappa = n - \dot{\omega}$ (the frequency of radial oscillations), and the Doppler-shifted azimuthal frequency, $n - n_p$. The particle will make a certain number of radial oscillations for every azimuthal oscillation.

e) What is the value of $x_{crit} > 0$ for which a trajectory at $x = x_{crit}$ just collides with a trajectory at $x = -x_{crit}$ (i.e., on the flip side of the resonance)? This is an estimate of the “width” of the resonance; it is an estimate of the width of the region near the edge of the planetary ring where perturbations by the shepherd satellite are greatest; within x_{crit} of a_0 , the velocity dispersion of ring particles can be substantially greater than the velocity dispersion of ring particles in the remainder of the ring that are well removed from the resonance.