

# Astro 250: Solutions to Problem Set 5

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## Problem 1. *The Titan ringlet: Part II*

*This problem follows on the heels of problem 1 in Problem Set 3.*

*The Colombo ringlet, also known informally as the Titan ringlet, is a narrow planetary ring around Saturn that sits near the 1:0 apsidal resonance established by the largest of the Saturnian moons, Titan. This means that the precession rate of the apsidal line of the ringlet is very close to the mean motion of Titan.*

*Denote Titan's mass over Saturn's mass by  $M_T/M = 2.366 \times 10^{-4}$ , its semi-major axis by  $a_T = 1.22 \times 10^6$  km, and its mean longitude by  $\lambda_T$ . Denote a single ring particle's semi-major axis by  $a = 77871$  km, its eccentricity by  $e = 2.6 \times 10^{-4}$ , and its mean motion by  $\Omega = 2.834 \times 10^{-4}$  rad/s.*

*In problem 1 of PS 3, one showed that for the ring particle near the resonance,*

$$\dot{a} = 0 \tag{1}$$

$$\dot{e} = -\eta \sin \phi \tag{2}$$

$$\dot{\tilde{\omega}} = -\frac{\eta}{e} \cos \phi \tag{3}$$

$$\phi \equiv \tilde{\omega} - \lambda_T \tag{4}$$

*where  $\eta = (M_T/M)\Omega\alpha H_{10}/2$ ,  $H_{10} = 2b_{1/2}^{(1)}(\alpha) + \alpha(d/d\alpha)b_{1/2}^{(1)}(\alpha) - 3\alpha$ , and  $\alpha = a/a_T$  are constants.*

*The above expression for  $\dot{\tilde{\omega}}$  is incomplete because it only accounts for mean-motion resonant forcing by Titan. What is missing is forcing by the local secular potential. Let's just say:*

$$\dot{\tilde{\omega}} = -\frac{\eta}{e} \cos \phi + \dot{\tilde{\omega}}_{sec} \tag{5}$$

*where  $\dot{\tilde{\omega}}_{sec} = \dot{\tilde{\omega}}_{Saturn} + \dot{\tilde{\omega}}_{stuff}$  is the total additional precession rate induced by the oblateness of Saturn and the secular potential of everything else—nearby rings (remember the Titan ringlet is just 1 narrow ring embedded in Saturn's gigantic ring complex), Titan, other satellites, the Sun, lost pens, etc. In fact, the first of these contributions completely overwhelms the other contributions. There is no need to write out explicitly what*

all these terms are; we will just work with  $\dot{\tilde{\omega}}_{sec}(a)$ . (Those of you who did problem 1 of PS 1 know what this function is, but the explicit form is not needed for this problem!)

Further define

$$\epsilon(a) \equiv \dot{\tilde{\omega}}_{sec}(a) - \Omega_T \quad (6)$$

and a special semi-major axis,  $a = a_0$ , such that  $\epsilon(a_0) = 0$ .

a) Based on the above, your notes, and/or problem 1 of PS 1, do you need to modify the equation for  $\dot{e}$  to account for the secular potential?

No,  $\dot{e}$  due to the oblateness of the planet is zero. For a ring particle in the plane, the potential presented by an oblate planet is time-independent and axisymmetric. Time-independence conserves energy, so that the semi-major axis of the particle is constant. Axisymmetry preserves the component of the angular momentum parallel to the axis of symmetry. In this case,  $\sqrt{a(1-e^2)}$  is conserved, and because  $a$  is conserved,  $e$  is conserved. (Note that  $e$  is not constant if the ring is out of the equator plane of the planet; we analyzed this behavior within the Kozai problem in class.)

b) Equations (2) and (5) are coupled ordinary differential equations. Replace  $e$  and  $\phi$  in favor of the variables,

$$h \equiv e \cos \phi \quad (7)$$

$$k \equiv e \sin \phi \quad (8)$$

Write down  $\dot{h}$  and  $\dot{k}$  in terms of  $\eta$ ,  $\epsilon$ ,  $h$ , and  $k$ .

Well,

$$e = \sqrt{h^2 + k^2} \quad (9)$$

$$\dot{e} = (h\dot{h} + k\dot{k})/e = -\eta \sin \phi \quad (10)$$

Also,

$$\dot{h} = -\dot{e} \cos \phi - e \sin \phi \dot{\phi} \quad (11)$$

$$= -\eta \sin \phi \cos \phi - k \dot{\phi} \quad (12)$$

Now  $\dot{\phi} = \dot{\tilde{\omega}} - \Omega_T = -(\eta/e) \cos \phi + \epsilon(a)$ . Insert this into (12) to find

$$\dot{h} = \eta \sin \phi \cos \phi - k[-(\eta/e) \cos \phi + \epsilon] \quad (13)$$

$$\boxed{\dot{h} = -\epsilon k} \quad (14)$$

Insert into (10) to find

$$\boxed{\dot{k} = -\eta + \epsilon h} \quad (15)$$

c) Solve your equations for  $\dot{h}$  and  $\dot{k}$ . Your solution should contain two arbitrary constants: an amplitude and a phase associated with a sinusoidal oscillation.

Take another time derivative of (14) and substitute for  $\dot{k}$  using (15):

$$\ddot{h} = -\epsilon\eta - \epsilon^2 h \quad (16)$$

The solution of this equation is

$$\boxed{h = e_{free} \cos(\epsilon t + \phi_{free}) + \eta/\epsilon} \quad (17)$$

where  $e_{free}$  and  $\phi_{free}$  are constants of integration. Substitute this solution into (15) and solve to find

$$\boxed{k = e_{free} \sin(\epsilon t + \phi_{free})} \quad (18)$$

d) Plot possible trajectories in  $h$  and  $k$  space. Identify the conditions under which  $\phi$  is circulating or librating. If the particle is librating, what are the libration centers,  $\langle \phi \rangle$ ? What libration centers are associated with  $a > a_0$ ? What centers are associated with  $a < a_0$ ? If you have the correct solution, you should notice that something terrible happens at  $a = a_0$ . This is simply a deficiency of our low-order theory.

I would make a postscript picture if I had time, but I don't so here goes: The trajectory in  $h$ - $k$  space is a circle whose center is located at  $(h = \eta/\epsilon, k = 0)$ , and whose radius is  $|e_{free}|$ .  $\boxed{\text{If } |e_{free}| > |\eta/\epsilon|, \text{ then the circle encloses the origin and } \phi \text{ circulates.}}$

$\boxed{\text{If } |e_{free}| < |\eta/\epsilon|, \text{ then the circle does not enclose the origin and } \phi \text{ librates.}}$  If  $a < a_0$ , then  $\epsilon(a) > 0$  (the secular precession rate increases with decreasing distance from the oblate planet), and the circle is traced out in the clockwise direction at angular speed

$\epsilon$ . The only possible libration center if  $a < a_0$  is  $\langle \phi \rangle = 0$ . All the signs are reversed if  $a > a_0$ , in which case the only possible libration center is  $\langle \phi \rangle = \pi$ .

Thus, if the ring particle is just outside the resonance ( $a > a_0$ ), then its pericenter is, on average, directed  $180^\circ$  away from Titan, whereas if it is just inside the resonance, then the pericenter, on average, points towards Titan. At exact resonance, our low-order theory explodes (it yields an infinite forced eccentricity,  $\eta/\epsilon$ ).

**Problem 2. Pluto and Neptune**

*Pluto inhabits two resonances: the 3:2 resonance for which  $\phi = 3\lambda_P - 2\lambda_N - \tilde{\omega}_P$  librates about  $\pi$  with an amplitude  $(\max(\phi) - \pi)$  of  $76^\circ$ , and the Kozai resonance for which  $\omega$  (the argument of perihelion) librates about  $\pi/2$  with an amplitude  $(\max(\omega) - \pi/2)$  of  $23^\circ$ . The inclination of Pluto's orbit to Neptune's is about  $16^\circ$ . The semi-major axis of Neptune is 30.1 AU and the semi-major axis of Pluto is 39.4 AU.*

a) *Pluto's eccentricity is 0.25. If this eccentricity grew from a primordial value of (say) 0.01 via resonant capture by a migrating Neptune and continued migration, what is the minimum distance over which Neptune must have migrated in semi-major axis? Give your answer in AU. Does your answer change substantially if Pluto's eccentricity were initially 0.05? 0.001?*

The adiabatic invariant associated with a  $p : p + q$  mean-motion resonance is  $C_{pq} = \sqrt{a}[(p+q) - p\sqrt{1 - e^2}]$ , where  $q < 0$  for exterior resonances (resonances that lie exterior to the perturber), all elements are those for the test particle (Pluto), and we are neglecting inclination effects. For the 3:2 resonance,  $p = 3$  and  $q = -1$ . Then  $C = \sqrt{a}[2 - 3\sqrt{1 - e^2}]$ . If  $e_i = 0.01$ ,  $e_f = 0.25$ , and  $a_f = 39.4$ , then

$$a_i = \frac{[3\sqrt{1 - e_f^2} - 2]^2}{[3\sqrt{1 - e_i^2} - 2]^2} a_f = 32.2 \tag{19}$$

At the onset of capture,  $(a_i/a_{N,i})^{3/2} = 3/2$ . Then  $a_{N,i} = 24.6$ . Then Neptune must have moved a hefty  $\Delta a_N = 5.5$  AU. This is a lower limit on  $\Delta a_N$  if we want resonant migration to also explain Pluto's hefty eccentricity of  $16^\circ$ . Currently fashionable estimates for  $\Delta a_N$  are near 7 AU.

If  $e_i = 0.05$ , then we calculate that  $\Delta a_N = 5.3$  AU. If  $e_i = 0.001$ , then we calculate that  $\Delta a_N = 5.5$  AU. Our answer is fairly insensitive to  $e_i$  if  $e_i \ll e_f$ .

b) *Estimate a lower limit to the timescale,  $a_N/\dot{a}_N$ , over which this migration occurred.*

By adiabatic invariance, for Pluto to remain caught in the 3:2 resonance, the timescale

over which the Hamiltonian is changing (i.e., the timescale over which Neptune is migrating) must exceed the natural oscillation period associated with the resonance, i.e., the resonant libration period. To order of magnitude, the resonant libration period of Pluto today is  $T_l \sim (2\pi/n_{Pluto})\sqrt{M_\odot/M_{Neptune}}/\sqrt{e_{Pluto}} \sim 7 \times 10^4 \text{ yr}$ . The actual answer from numerical integrations is  $1.9 \times 10^4 \text{ yr}$ . In the past,  $n_{Pluto}$  must have been larger by a factor of a few, and its eccentricity must also have been smaller; both these factors tend to cancel out in our expression for the libration timescale. Thus, we estimate that  $a_N/\dot{a}_N \gg 10^4 \text{ yr}$ . Numerical simulations of Neptune's migration bear this estimate out [see Chiang & Jordan (2002)].

*c) What is the closest distance of approach that Pluto makes to Neptune? Recognize that the libration period of the 3:2 resonance is much shorter than the libration period within the Kozai resonance, so that you don't need to worry about commensurabilities. Compare your answer to the Hill sphere radius of Neptune.*

Doing this problem rigorously would require a straightforward Monte Carlo survey over all possible orbital elements of Pluto and Neptune. We will satisfy ourselves with an estimate based on qualitative spatial reasoning. The closest approach that Pluto makes to Neptune will occur when Pluto is near perihelion, so let us set the mean anomaly of Pluto  $M_P = 0 = f_P$ . The 3:2 resonance angle  $\phi$  librates from  $180 - 76$  degrees to  $180 + 76$  degrees; since in exact resonance  $\phi = 180$  degrees and Pluto is most well removed from Neptune, it follows that Pluto is least well removed from Neptune if  $\phi$  is at furthest excursion from exact resonance, that is  $= 180^\circ - 76^\circ = 104^\circ$ . (We could equally well have chosen the  $180^\circ + 76^\circ$ .) Without loss of generality, say that  $\lambda_N = 0$  and use  $\phi = 104^\circ$  to find that  $\tilde{\omega}_P = 52^\circ$ . Similar reasoning applied to the Kozai libration tells us that Pluto will be least well removed from Neptune if  $\omega_P = 90^\circ - 23^\circ = 67^\circ$ . Then the longitude of ascending node,  $\Omega_P = \tilde{\omega}_P - \omega_P = -15^\circ$ . The remaining elements are  $a_P = 39.4 \text{ AU}$ ,  $e_P = 0.25$ , and  $i_P = 16^\circ$ . In reality,  $e_P$  and  $i_P$  vary slightly over the 3:2 and Kozai oscillations, but we will neglect these variations here.

Now use the coordinate transformation equations (2.122) in Murray & Dermott to solve for the Cartesian coordinates of Neptune and Pluto. For Neptune,  $(x, y, z) = (30.1, 0, 0)$ . For Pluto,  $(17.1, 22.8, 7.5)$ . Our estimate for the minimum distance between Neptune and Pluto is then  $\boxed{27 \text{ AU}}$ . The actual answer is closer to  $\boxed{20 \text{ AU}}$ . Probably our greatest source of error is in our assumption that Pluto and Neptune are closest when Pluto is at perihelion. In any case, the minimum approach distance is much greater than the  $\boxed{\text{Hill radius of Neptune, } a_N(m_N/3M_\odot)^{1/3} = 0.77 \text{ AU}}$ . Thus, Pluto is never in any danger of suffering a close encounter with Neptune.

**Problem 3.** *N petals, forced eccentricities, and another definition of a resonant width*

*This problem generalizes problem 1 of this set. It is relevant for the resonant edges of planetary rings.*

The edges of planetary rings are near principal Lindblad resonances of azimuthal wavenumber  $m$  established by shepherd satellites. At the exact resonance location,

$$(m \mp 1)n - mn_p \pm \dot{\tilde{\omega}} = 0. \quad (20)$$

Here  $m$  is a positive integer,  $n$  and  $n_p$  are the mean motions of a ring (test) particle and of the perturbing shepherd, and  $\dot{\tilde{\omega}}$  is the apsidal precession rate of the ring particle. The upper/lower signs correspond to inner/outer Lindblad resonances.

Take the shepherd to be outside the ring. The resonant disturbing function of the shepherd is

$$R_{p,res} = \frac{Gm_p}{a_p} f(\alpha) e \cos \phi \quad (21)$$

$$\phi = (m - 1)\lambda - m\lambda_p + \tilde{\omega} \quad (22)$$

where  $\lambda$ 's are mean longitudes,  $e$  is the eccentricity of the test particle, and  $f(\alpha) = f(a/a_p)$  is a dimensionless function of the ratio of semi-major axes of the particle to the perturber.  $f$  is often of order unity.

a) Calculate  $\dot{\tilde{\omega}}_{res}$  and  $\dot{e}_{res}$  from  $R_{p,res}$  using Lagrange's planetary equations. (We are neglecting the variation in semi-major axis because we emphasized in lecture that the fractional variation in  $a$  is  $e^2$  smaller than the fractional variation in  $e$ .)

Lagrange's equations to leading order in eccentricity read,  $\dot{e} = (-1/na^2e)\partial R/\partial\tilde{\omega}$  and  $\dot{\tilde{\omega}} = (+1/na^2e)\partial R/\partial e$ . These give

$$\dot{e} = \frac{m_p}{m_c} \frac{af(\alpha)}{a_p} n \sin \phi \quad (23)$$

$$\dot{\tilde{\omega}} = \frac{m_p}{m_c} \frac{af(\alpha)}{a_p} \frac{1}{e} n \cos \phi \quad (24)$$

where  $m_c$  is the mass of the central object (planet).

b) It is evident that  $\dot{\phi} = (m - 1)n - mn_p + \dot{\tilde{\omega}}$ . In reality,  $\dot{\tilde{\omega}} = \dot{\tilde{\omega}}_{res} + \dot{\tilde{\omega}}_{sec}$ . For this problem, we will consider  $m \neq 1$  and say that  $\dot{\tilde{\omega}}_{sec} \ll \dot{\tilde{\omega}}_{res}$ . Note that we cannot ignore  $\dot{\tilde{\omega}}_{sec}$  if  $m = 1$ ; see problem 1 above. Many planetary rings have their edges located at  $m \sim 10$ .

Similarly ignore  $\dot{e}_{sec}$ .

Define  $\epsilon(a) = (m - 1)n - mn_p$  to write

$$\dot{\phi} = \epsilon(a) + \dot{\omega}_{res} \quad (25)$$

Now take the particle to be firmly in the resonance with vanishingly small libration amplitude; that is, consider the limit  $\dot{e} \rightarrow 0$  and  $\dot{\phi} \rightarrow 0$ . What are the equilibrium values for  $e$  and  $\phi$ ? The value for  $e$  that you have deduced is called the “forced eccentricity” (as opposed to the “free eccentricity,” which is the amplitude of libration in  $h$ - $k$  space; see problem 1 above). Remember that  $\epsilon(a)$  can be either negative or positive, so you should never get a negative eccentricity.

$\dot{e} = 0$  demands  $\boxed{\phi = 0, \pi}$ .  $\dot{\phi} = 0$  demands  $\dot{\omega}_{res} = -\epsilon(a)$ . Recognize that our answer is part (a) is actually  $\dot{\omega}_{res}$ . Then use (24) to solve for  $\boxed{e = |(m_p a f(\alpha)/m_c a_p)(n/\epsilon)|}$ , where we have not bothered to worry about the sign of  $\epsilon$  and the sign of  $\cos \phi$ .

c) Express the eccentricity  $e$  in terms of the distance,  $x = a - a_0$ , where  $(m - 1)n(a_0) = mn_p$ . Of course, we are considering  $x \ll a_0$ .

Write  $n = [mn_p/(m - 1)](1 + x/a_0)^{-3/2} \approx [mn_p/(m - 1)](1 - 3x/2a_0)$  and insert into part (c) to find

$$\boxed{e = \left| \frac{m_p}{m_c} \sqrt{\frac{a_p}{a_0}} \frac{f(\alpha)}{m} \frac{2a_0}{3x} \right|} \quad (26)$$

d) In the frame co-rotating with the shepherd (which we take to be moving on a perfectly circular orbit), SKETCH APPROXIMATELY the trajectories of ring particles for a few values of  $x$ , both positive and negative. You may find it helpful to think in terms of epicyclic frequency,  $\kappa = n - \dot{\omega}$  (the frequency of radial oscillations), and the Doppler-shifted azimuthal frequency,  $n - n_p$ . The particle will make a certain number of radial oscillations for every azimuthal oscillation.

For a resonant particle with zero libration amplitude,  $\dot{\phi} = 0 = (m - 1)n - mn_p + \dot{\omega}$ . The epicyclic frequency  $\kappa = n - \dot{\omega} = n + (m - 1)n - mn_p = m(n - n_p)$ . Then in one synodic period,  $2\pi/(n - n_p)$ , the particle completes  $m$  radial oscillations. The amplitude of the radial oscillation, i.e., the maximum radial deviation from the guiding circle, is  $ea$ .

Go into the frame rotating with the mean motion of the shepherd, and have the shepherd’s (fixed) position be on the  $x$ -axis in this frame. Then the ring particle will trace out an  $m$ -petalled pattern in this frame; i.e. a flower with  $m$  number of petals. When the ring particle achieves conjunction with the shepherd, whether the particle is at its apoapse or at its periapse depends on whether  $a < a_0$  or  $a > a_0$ . If  $a > a_0$ , then  $\epsilon < 0$  which in turn implies that the libration center  $\phi = 0$ . If  $\phi = 0$ , then at conjunction

( $\lambda = \lambda_p$ ),  $\lambda = \tilde{\omega}$ —the particle is at its periapse. Thus, at  $a > a_0$ , we orient an  $m$ -petalled flower (or an  $m$ -toothed gear, if you like) such that a trough lies on the x-axis. If  $a < a_0$ , we orient the  $m$ -petalled flower such that a crest lies on the x-axis. The height of the petals decreases as the distance from exact resonance increases; in other words, as  $|x|$  increases,  $ea$  decreases.

*e) What is the value of  $x_{crit} > 0$  for which a trajectory at  $x = x_{crit}$  just collides with a trajectory at  $x = -x_{crit}$  (i.e., on the flip side of the resonance)? This is an estimate of the “width” of the resonance; it is an estimate of the width of the region near the edge of the planetary ring where perturbations by the shepherd satellite are greatest; within  $x_{crit}$  of  $a_0$ , the velocity dispersion of ring particles can be substantially greater than the velocity dispersion of ring particles in the remainder of the ring that are well removed from the resonance.*

The amplitude of each petal is  $ea_0$ ; a petal at  $x < 0$  just touches the petal at  $x > 0$  when  $ea_0 = x = x_{crit}$ . Insert (26) into this equation to find

$$\boxed{x_{crit} = \sqrt{\frac{m_p}{m_c} \frac{2f(\alpha)}{3m} \left(\frac{a_p}{a_0}\right)^{1/2}} a_0} \quad (27)$$

To order of magnitude,  $x_{crit} \approx \sqrt{m_p/m_c} a_0$ . For satellites shepherding the  $\epsilon$  ring of Uranus, this distance is a few km. Thus, in the  $\epsilon$  ring which measures  $\sim 60$  km radially, you should imagine the last few km near either resonant edge being stirred up dramatically by the shepherds.