Astro 250: Solutions to Problem Set 6

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Problem 1. *Hot Jupiters*

About 10 extrasolar, Jovian-mass planets have been discovered in extremely close proximity to their parent, solar-type stars. They have semi-major axes of about 0.05 AU and low eccentricities < 0.02. *They have been referred to as "hot Jupiters."*

a) Consider a Jupiter-mass planet in an orbit of small eccentricity about a solar-mass star. Calculate the semi-major axis, a, inside of which the eccentricity damping timescale, $e/\dot{e}$, *is less than* 10^9 yr *(of order the age of the star). Assume that the planet's* $Q \sim 10^5$ *and that* $k_2 \sim 0.5$. *Is it surprising that hot Jupiters have low eccentricity?*

Use equation (4.198) to solve for a

$$\boxed{a = \left[\frac{63\tau_e}{4} \frac{M_\odot}{m_J} \frac{r_J^5 \sqrt{GM_\odot}}{\tilde{\mu} Q_J} \right]^{2/13}} \tag{1}$$

where $\tau_e = e/\dot{e}$. Take the dimensionless effective rigidity to be of order unity, $\tilde{\mu} \sim 3/2k_2 \sim 3$. Plugging in, we find, $\boxed{a = 0.056 \text{ AU}}$. Thus, for the hottest hot Jupiters, it is not surprising that their orbital eccentricities are unmeasurably small.

b) Astronomers have attempted to detect rings or satellites around HD209458b, a hot Jupiter whose orbit crosses the face of its parent star to generate a detectable, periodic decrease in the stellar intensity; the planet transits the star. The shape of the stellar light curve (intensity vs. time) has been analyzed for the presence of rings or satellites (so far, no luck).

Consider a shepherd moon of a ring system orbiting HD209458b. What is the timescale for its semi-major axis evolution due to tidal interaction, $a_s/\dot{a}_s$*? Is it longer than the age of the star? Take the shepherd's initial semi-major axis to be* $a_s = 2$ *planetary radii (inside the Roche zone—see problem 3), the shepherd's size to be* $R_s = 10\,\text{km}$ *(comparable to the sizes of known shepherds in the solar system), and HD209458b to be identical to Jupiter. First decide whether the shepherd's orbit will decay or expand by making a reasonable assumption about the spin period of HD209458b.*

Let's assume that the planet is caught in a low-order spin-orbit resonance, so that its spin period is of order its orbit period, which equals 3 days. A shepherd at $a_s = 2 \times 7 \times 10^9$ cm has an orbital period of 4.4×10^3 s ≈ 1 hr; thus, it lives below synchronous orbit so its orbit will decay. The timescale for orbital decay is given by equation (4.160):

$$\frac{a_s}{\dot{a}_s} = \frac{Q_p}{3k_{2p}} \frac{m_p}{m_s} \left(\frac{a_s}{R_p}\right)^5 \frac{1}{n} \tag{2}$$

Plugging in numbers, we find $\boxed{\frac{a_s}{\dot{a}_s} \sim 10^{13}\,\mathrm{yr}}$, which is several hundred times the age of the star.

Problem 2. *Stability of Double Synchronous State*

Consider an isolated planet-satellite system in which the planet's spin angular momentum is parallel to the orbital angular momentum. Take the planet to have a constant axial moment of inertial, I, and approximate the satellite by a point mass. Tidal interactions conserve the total angular momentum, L, but decrease the total (potential plus kinetic) mechanical energy, E, of this system. Denote the radius of the relative orbit by a and the reduced mass by $m = m_p m_s/(m_p + m_s)$ where m_p and m_s are the masses of the planet and satellite.

(a) Write down equations for L and E in terms of m_p, m_s, m, the planet's spin rate, Ω, and the orbital frequency, n.

The energy comprises the spin kinetic energy of the satellite, plus the total energy of the bound gravitational system:

$$E = \frac{1}{2} I \Omega^2 - \frac{G m_s m_p}{2a} \tag{3}$$

The semi-major axis $a^3 = G(m_s + m_p)/n^2$ by Kepler, so that:

$$\boxed{E = \tfrac{1}{2} I \Omega^2 - \tfrac{1}{2} (G m_s m_p)^{2/3} m^{1/3} n^{2/3}} \tag{4}$$

Measure the spin + orbital angular momentum in the inertial frame, with the central axis passing through the center of mass:

$$L = I\Omega + m a^2 n \tag{5}$$

Replace a with n to write:

$$\boxed{L = I\Omega + (G m_p m_s)^{2/3} m^{1/3} n^{-1/3}} \tag{6}$$

(b) Show that together $\dot{L} = 0$ and $\dot{E} < 0$ imply

$$(n - \Omega)\,\dot{\Omega} > 0 \quad \text{and} \quad (n - \Omega)\,\dot{n} > 0.$$

Thus if $n > \Omega$, the orbit shrinks and the planet spins up, whereas if $n < \Omega$, the orbit expands and the planet spins down.

Let $C = (Gm_p m_s)^{2/3} m^{1/3}$. Then $\dot{L} = 0$ implies:

$$I\dot{\Omega} = \frac{1}{3} C n^{-4/3} \dot{n} \tag{7}$$

By $\dot{E} < 0$ we have:

$$I\Omega\dot{\Omega} - \frac{1}{3} C n^{-1/3} \dot{n} < 0 \tag{8}$$

Substitute the former equation for $\dot{\Omega}$ into the latter:

$$\begin{aligned} \frac{1}{3} C n^{-4/3} \dot{n} \Omega - \frac{1}{3} C n^{-1/3} \dot{n} &< 0 \\ \implies (n - \Omega)\dot{n} &> 0 \end{aligned} \tag{9}$$

Of course, we could have substituted for $\dot{n}$ instead of $\dot{\Omega}$. Then we would have found:

$$(n - \Omega)\dot{\Omega} > 0 \tag{10}$$

c) Derive the expression

$$\frac{\dot{\Omega}}{\dot{n}} = \frac{ma^2}{3I}.$$

Use (7) above to write:

$$\begin{aligned} \frac{\dot{\Omega}}{\dot{n}} &= \frac{C n^{-4/3}}{3I} \\ &= \frac{ma^2}{3I} \end{aligned} \tag{11}$$

where the last equality follows from Kepler and from our definition of C.

d) What parameter determines the stability of the state in which $\Omega = n$? Estimate the value of this parameter for the Pluto-Charon system.

If $n > \Omega$, then both $\dot{\Omega}, \dot{n} > 0$. To return back to the state where $\Omega = n$, we must have $\dot{\Omega} > \dot{n}$ (so that Ω catches back up to n eventually). By (c), we see that such stability requires

$$\boxed{\Gamma \equiv \frac{ma^2}{3I} > 1} \tag{12}$$

The same criterion results if we consider $n < \Omega$.

For Pluto-Charon, $a = 19636\,\text{km}$, $m_p \approx 0.003 M_\oplus$, $m_c \approx 0.0004 M_\oplus$, $m \approx 0.0004 M_\oplus$, and $I \approx \frac{2}{5} m_p R_p^2$, where $R_p \approx 1137\,\text{km}$. Then $\boxed{\Gamma \approx 40}$ and we see that the Pluto-Charon system is $\boxed{\text{stable}}$.

Problem 3. *Tidal Disruption and the Roche Zone*

This problem examines why ring systems about all the giant planets occupy planetocentric distances that are less than ~2 planetary radii.

a) Consider a perfectly rigid, spherical satellite of radius R_s, mass m_s, and density ρ_s orbiting a planet of radius R_p, mass m_p, and density ρ_p. Assume the satellite to be in synchronous lock, so that its spin period matches its orbital period. Take the satellite's orbital semi-major axis to be a_s and its orbital eccentricity to be zero.

A marble rests on the surface of this spinning satellite. The spin of the satellite tries to spin it off. The tidal field of the planet also tries to pull it off. The only force trying to keep it glued to the satellite is the satellite's own gravity. For small enough a_s, the marble will fly off. What is this minimum semi-major axis, $a_{s,1}$? Express in terms of ρ_s, ρ_p, and R_p.

The marble is at its most unstable when it lies right in between the planet and the host satellite. Then the tidal acceleration from the planet acting to pull the marble off is $|(d/da)(Gm_p/a^2)R_s| = 2(Gm_p/a^3)R_s$. The centrifugal acceleration of the satellite acting to spin the marble off is $(Gm_p/a^3)R_s$. At $a = a_{s,1}$, these accelerations add to barely balance the satellite's gravitational pull, Gm_s/R_s^2. Then

$$3\frac{Gm_p}{a_{s,1}^3}R_s = \frac{Gm_s}{R_s^2} \tag{13}$$

Solve for

$$\boxed{a_{s,1} = \left(\frac{3\rho_p}{\rho_s}\right)^{1/3} R_p = 1.44 \left(\frac{\rho_p}{\rho_s}\right)^{1/3} R_p} \tag{14}$$

b) How does your answer in (a) relate to the radius of the Hill sphere of the satellite, $r_H = (m_s/3m_p)^{1/3} a_s$?

$\boxed{\text{When } a = a_{s,1},}$ then

$$r_H = (\rho_s/3\rho_p)^{1/3}(R_s/R_p)(3\rho_p/\rho_s)^{1/3} R_p = R_s$$

i.e., $\boxed{\text{the satellite just fills its Hill sphere (Roche lobe)}}$. When $a < a_{s,1}$, we say the satellite overfills its Roche lobe. When $a > a_{s,1}$, we say the satellite underfills its Roche lobe.

c) Now consider a marble floating on a perfectly strengthless, fluid, synchronously rotating satellite. The satellite's shape is now free to distort because it is sitting in the tidal field of the planet and because it is spinning.

ESTIMATE (no heroics necessary) the semi-major axis of the satellite, $a_{s,2}$, inside of which the marble flies off the watery satellite. This is a repeat of (a) except that you will need to account for the distorted shape of the satellite; the satellite has an enhanced size due not only to the tide raised on it by the planet, but also due to its spin. You should at least decide whether $a_{s,2}$ should be larger or smaller than $a_{s,1}$.

The marble now floats an extra distance away from the center of the satellite. Let's estimate the new distance as $R' = R_s[1 + (m_p/m_s)(R_s/a)^3 + \omega^2/G\rho_s]$, where the first enhancement factor comes from the tide raised on the watery satellite by the planet, and the second enhancement factor comes from the spin-induced bulge. For a synchronous satellite, the spin $\omega^2 = Gm_p/a^3$. Then $R' = R_s\{1 + [(4\pi + 3)/3](m_p/m_s)(R_s/a)^3\}$. Replace R_s in (13) by R' to find

$$3x\left(1 + \frac{4\pi + 3}{3}x\right)^3 = 1 \tag{15}$$

where $x \equiv (m_p/m_s)(R_s/a)^3$. My pocket calculator gives $x \approx 0.0995$. Then

$$a_{s,2} = x^{-1/3}\left(\frac{m_p}{m_s}\right)^{1/3} R_s \tag{16}$$

$$\boxed{a_{s,2} = 2.2\left(\frac{\rho_p}{\rho_s}\right)^{1/3} R_p} \tag{17}$$

Chandrasekhar did a more proper job and got the coefficient to be 2.46 instead of our 2.2.

d) At orbital semi-major axes of less than $\sim$*2 planetary radii, there are no satellites whose sizes exceed 100 km, but there are rings composed of meter-sized boulders and smaller debris, and 10 km-sized satellites. Given your answers in (a) and (c), explain why these observations make sense.*

Since $\rho_p/\rho_s \approx 1$, we should expect bodies which are held together mostly by self-gravity to be ripped apart by *both* tidal and centrifugal forces if they are found at planetocentric distances less than $[1.4\text{–}2.2]R_p$. Small bodies which are held together mostly by intermolecular cohesive forces rather than self-gravity are immune to this disruption. Thus, plenty of small bodies—ring particles, small satellites—can exist within $\sim 2R_p$, but large bodies are torn apart. The tidal/centrifugal disruption zone is referred to as the "Roche zone."