Astrophysical Fluid Dynamics – Problem Set 1

Readings: Shu pages 3–6; Tritton 5.1–5.5; White 4.2, 4.3, 4.5; Pringle & King Sections 1 through 1.4, and 1.7

Problem 1. Scaling the Heights

(a) Write down an expression for the variation of gas density ρ (units of mass per volume) with height z above the Earth's surface, assuming the gas is at constant temperature T and in hydrostatic equilibrium. Take the gas to be made of a single kind of molecule of weight μ (units of mass). Express in terms of the density at ground level $\rho_0 \equiv \rho(z=0)$ and the *density scale height* $h \equiv kT/(\mu g)$, where g is the gravitational acceleration. Neglect variations of g with z.

(b) Utterly neglecting collisions with other molecules, what is the maximum height a molecule would attain if launched from $z = 0$ with a typical thermal velocity? Is this close to h?

This calculation is misleading insofar as it ignores collisions between molecules, which are crucial for describing air as a continuum fluid. Nevertheless, it provides a mnemonic for remembering the scale height, and it illustrates the sometimes surprisingly close connection between fluid mechanics and particle mechanics (kinetic theory).

(c) At what height z would you expect the formula in (a), which depends on the continuum hypothesis, to fail? This height marks the location of the *exobase* in planetary atmospheres. Just consider how intermolecular collisions validate the continuum approximation.

(d) Write down an expression for the hydrostatic variation of gas density ρ with height z above the midplane of a circumstellar disk at radius r. As in (a), assume constant T and μ. Take the gravitational field to be that from the star alone (ignore the self-gravity of the disk). Work in the limit that $z \ll r$. Express in terms of the density at the midplane ρ_0 and the density scale height $h \equiv (kT/\mu)^{1/2}\Omega^{-1}$, where Ω is the Keplerian orbital angular frequency.

The height h is often written $h = c_s/\Omega$, where $c_s = (kT/\mu)^{1/2}$ is the speed of sound waves in gas that behaves isothermally [by behaving isothermally, we mean that the gas keeps the same temperature regardless of how it is compressed or expanded by the sound wave. This problem does not concern a sound wave *per se*, but people talk about the sound speed anyway because sound waves (pressure disturbances) are the means by which a medium establishes hydrostatic equilibrium.]

(e) As in (b), calculate the maximum height that a gas molecule would attain if launched upwards from the midplane with a typical thermal speed, ignoring collisions. Compare to h.

(f) For the disk to be geometrically "thin" ($h/r \ll 1$), what must be true about $c_s/v_{\rm K}$? Here $v_{\rm K}$ is the Keplerian orbital velocity.

Problem 2. Tip of the Iceberg

An iceberg is sighted with volume V above the water's surface. What volume of ice must reside below the water's surface? Take the density of berg ice, which is composed of frozen freshwater and full of bubbles, to be $\rho_{\rm ice} = 0.9\,{\rm g\,cm^{-3}}$, and the density of seawater to be $\rho_{\rm sea} = 1.03\,{\rm g\,cm^{-3}}$.

Try not to merely invoke Archimedes' principle, but instead try to derive the answer from hydrostatic equilibrium.

OPTIONAL Problem 3. To d/dt or Not to d/dt

Give a physical explanation for why the left-hand side of the momentum equation reads $\rho du/dt$ and not $d(\rho u)/dt$, where u is velocity, ρ is density, and d/dt is the convective derivative. What is being assumed about the fluid parcel in the Lagrangian view?