Astrophysical Fluid Dynamics – Problem Set 11

Readings: Course Reader: (a) selections from Binney & Tremaine; (b) passages on accretion disks from Frank, King, and Raine; (c) however much of Gammie (2001) that you like (the introduction is very readable). Also skim Shu 82–88, 90–92.

Problem 1. Toomre Q

A gas disk orbits an object of mass M. The disk is in vertical hydrostatic equilibrium.

(a) Show that $Q < 1$ at radius r corresponds to the condition $\rho > \rho_{\rm crit} \sim M/r^3$, where ρ is the gas density (mass per unit volume) at the disk midplane. The condition you have derived is an easy way to remember the Toomre instability condition. The critical density $\rho_{\rm crit}$ is sometimes called the Roche density (and is just the mass of the central object divided into the sphere just enclosing your position).

(b) Show that for $Q \sim 1$, the marginally unstable mode has a radial wavelength comparable to the gas vertical scale height h.

(c) Papers will occasionally state that $Q < 1$ corresponds to $M_{\rm disk}/M > h/r$. Explain why this statement makes (rough) sense.

Problem 2. Thank You Sir May I Have Another

Consider spiral disturbances of the form $\exp i(k_r r - \omega t + m\phi)$ in a disk. Note that $k_r < 0$ for leading waves and $k_r > 0$ for trailing waves.

This problem explores basic kinematics of waves in disks. By solving part (c), you will see why the WKB dispersion relation that we presented in class cannot be used to study the stability of non-axisymmetric ($m \neq 0$) waves, because that dispersion relation assumes that waves always remain tightly wound and that k is constant.

(a) Show that $\cot i = |k_r r/m|$, where i is the pitch angle between the tangent to a spiral arm and the local circle $r =$ constant.

This formula is valid for any value of i. In the special case that waves are tightly wound (permitting a WKB analysis), $i \ll 1$ and $|kr| \gg 1$.

(b) Consider a parcel of gas on a purely circular orbit with angular frequency Ω. With what frequency do the spiral arms wash over this parcel of gas, for $m \neq 0$?

This frequency is sometimes called the Doppler-shifted forcing frequency in the disk literature ("Doppler-shifted" to remind you that everything is in relative circular motion, and "forcing" to remind you that a given gas parcel is being kicked by the spiral disturbance at this frequency.)

(c) At the so-called "corotation circle," the parcel does not get kicked by the wave, because it is moving at just the right angular frequency Ω to keep up with the wave (the parcel is "corotating" with the wave). Every wave has its own corotation circle.

Consider a non-axisymmetric wave ($m \neq 0$) at whose corotation circle $r = R_0$ and $\Omega = \Omega_0$. Go into the frame that corotates with the wave. In a differentially rotating disk, the wavefronts get turned around by the background shear. In other words, leading waves become trailing waves at the corotation circle.

Derive expressions for $k_r(t)$ and $k_\phi(t) = 2\pi/\lambda_\phi(t)$ (the wavenumber in the azimuthal direction), in terms of the initial value $k_r(0)$, m, R_0, Ω_0, and t. Restrict the analysis to Kepler shear ($\Omega \propto r^{-3/2}$) and to small radial displacements about corotation ($\Delta r \ll r$). The latter restriction constitutes the so-called "shearing sheet" approximation of disk dynamics. In the shearing sheet, the flow appears Cartesian (like the viscous flow between two parallel plates that you explored in a previous problem). You should find the wavefronts remaining straight while they turn from being leading to trailing. Do waves that are initially leading and tightly wound remain tightly wound?

Problem 3. Alpha Disks

Consider a disk having a dimensionless viscosity α. The disk accretes at a steady rate $\dot{M}$.

The disk cools radiatively. Neglect the difference between the effective temperature of the disk $T_{\rm eff}$ (which is nothing more than a convenient way of stating what the emitted flux is) and the actual gas kinetic temperature T. Take the gas to have sound speed c_s and angular frequency Ω, both of which vary with disk radius r.

(a) Find how h/r scales with r, where h is the disk vertical scale height. Sketch how the disk looks.

(b) Find how σ scales with r.

(c) Find how the disk midplane density ρ scales with r.

(d) Find an approximate expression for how long it takes a pressure disturbance to equilibrate away. Call this time t_z, and express it as simply as possible.

(e) Find an approximate expression for how long it takes a temperature disturbance to equilibrate away. Call this time $t_{\rm cool}$, and express in terms of Ω and α.

(f) Find an approximate expression for how long it takes a mass disturbance (say, a local bunching of material) to viscously diffuse away. Call this time $t_{\rm visc}$, and express in terms of Ω, α, and h/r.

Arrange t_z, $t_{\rm cool}$ and $t_{\rm visc}$ in increasing order.

(g) Find an expression for the critical radius $r_{\rm crit}$ beyond which Toomre's $Q < 1$. Alpha disks are generically unstable at large radii, leading some to surmise that the outer peripheries of quasar accretion disks / protostellar accretion disks are fertile breeding grounds for starbursts / binary companion stars or brown dwarfs.