Astrophysical Fluid Dynamics – Problem Set 3

Readings: Tritton, Chapter 3; Movies: Drag I, II, III, IV; relevant pages on drag in the Course Reader

Note: Many of the problems below are order-of-magnitude problems where we are only interested in ballpark answers accurate to within factors of a few. Feel free to make simplifying assumptions while justifying those assumptions when possible.

Problem 1. Newtonian viscous flow between two infinite plates

Consider two infinite flat plates oriented parallel to one another in the x-direction and separated by distance L in the y-direction. The space between the plates is filled with a Newtonian fluid having constant density ρ and kinematic viscosity ν. At $t < 0$, the fluid and the plates are initially at rest. At $t > 0$, the upper plate located at $y = +L$ moves in the x-direction at constant velocity u_0. The lower plate at $y = 0$ remains at rest for all time.

(a) [5 points] What is the steady-state flow? Write down the steady velocity field of the fluid and show that it gives a time-independent solution to the momentum equation.

(b) [5 points] The steady state you have written down in part (a) is attained at "late times", after all of the fluid has had time to respond to the upper plate which was suddenly set in motion. Sketch what the flow looks like at intermediate times, on the approach to steady state, starting at $t = 0$. You may draw pictures, sketch plots, or both. ALSO give an order-of-magnitude formula for the time T it takes the flow to relax to the steady state.[1]

Problem 2. Equation of state for an ideal gas

The first law of thermodynamics states that a system of volume V, entropy S, internal energy E, pressure P, and temperature T behaves according to

$$dE = -P\,dV + T\,dS \tag{1}$$

where by construction

$$P \equiv -\left(\frac{\partial E}{\partial V}\right)_S . \tag{2}$$

The partial derivative above is performed at constant entropy.[2]

Consider the specific example of an ideal gas of fixed total mass M, inside a (changeable) volume V. The ratio of specific heats at constant pressure and constant volume is

[1] Formally it takes infinite time to reach the precise steady state in part (a). However, the flow "resembles" the steady state after the time you wrote down in part (b), i.e., after time T, the flow matches the steady state to within a factor of a few.

[2] An analogous definition can be given for temperature: $T \equiv (\partial E/\partial S)_V$.

$\gamma \equiv C_P/C_V$. Recall that an ideal gas at constant entropy obeys the adiabatic relation $P = K\rho^\gamma$ for some constant K.[3]

[5 points] Integrate the first law of thermodynamics (1) at constant entropy to derive an expression for the pressure P:

$$P = \rho\varepsilon(\gamma - 1) \tag{3}$$

where $\rho \equiv M/V$ is the mass density and $\varepsilon \equiv E/M$ is the specific internal energy.

The equation of state (3) that you have derived is used routinely in astrophysical fluid calculations. Although it is derived by assuming constant entropy according to the definition (2), it applies generally, including in non-adiabatic situations which do not conserve entropy.

Problem 3. LOST

Desmond turns the key and your plane splits in half at an altitude of 32,000 feet above the Pacific Ocean. You are not wearing your seatbelt and go flying out.

[5 points] About how long does it take for you to hit the water's surface?

Problem 4. Particle Drift in Protoplanetary Disks

Planets form in disks of gas and dust surrounding young stars.

Consider the young Sun, encircled by a disk of gas of surface density (mass per unit face-on area) $\Sigma \approx 10^3\,\mathrm{g\,cm^{-2}}$ in the vicinity of 1 AU. The temperature of disk gas is $T \approx T_0(a/\,\mathrm{AU})^{-q}$, where $T_0 \approx 200\,\mathrm{K}$, a is the disk radius, and $q \approx 1/2$. Call the corresponding sound speed $c_s \approx \sqrt{kT/\mu}$, where k is Boltzmann's constant and μ is the mean molecular weight.

(a) [5 points] Consider a dust particle of size $s = 1\,\mu\mathrm{m}$ and internal density $\rho_p \approx 1\,\mathrm{g\,cm^{-3}}$ located one scale height above the disk midplane, $z = h$ (see Problem Set 1). Estimate the time the particle takes to settle *vertically* from $z = h$ to $z = h/2$. We call this a characteristic settling time for dust. Provide both a symbolic answer and a numerical answer in years.

(b) [5 points] Does disk gas rotate at sub-Keplerian or super-Keplerian speeds? Estimate the difference Δv between the gas velocity and the circular Keplerian speed $v_K \equiv \Omega a \equiv \sqrt{GM_\odot/a}$, at the midplane $z = 0$ at $a = 1$ AU. Express Δv symbolically in terms of c_s, Ω, a, and whatever dimensionless numbers you need. Also provide a numerical estimate. Hint: Consider the radial pressure gradient.

(c) [5 points] Particles (rocks) are considered "well-entrained" in disk gas—i.e., they rotate with nearly the same velocity as disk gas—if their *momentum stopping time*

[3] The constant K measures the system's specific entropy.

$$t_{\text{stop}} \equiv \frac{m v_{\text{rel}}}{F_D} \tag{4}$$

is short compared to the *dynamical time* $t_{\text{dyn}} \sim 1/\Omega$ [this latter time is the time for gas to change its velocity by $\mathcal{O}(1)$ (an order-unity factor)—in the case of gas in rotation, the time it takes for the disk velocity to change its direction by $\mathcal{O}(1)$ radian]. Here m is the particle mass, v_{rel} is the relative velocity between a particle and disk gas, and F_D is the drag force on the particle.

Estimate the critical size s_{crit} for which $t_{\text{stop}} \sim t_{\text{dyn}}$ at the midplane $z = 0$ at $a = 1$ AU. A numerical answer suffices. Are bodies larger or smaller than s_{crit} well-entrained?

(d) [5 points] About how long does it take a critically sized particle to drift *radially* inwards due to drag? Assume such a particle rotates at the full Keplerian speed. Then from part (b), one knows that this particle experiences a drag force which decreases its orbital angular momentum. In a Keplerian potential, if a particle decreases its angular momentum, its orbit shrinks. The particle starts at $z = 0$ at $a = 1$ AU, and we want to know how long it takes to reach, say, $a = 0.5$ AU. A numerical estimate suffices. (Hint: the drag law that applies in part (a) does not necessarily apply here in part (d), because we are talking about a different size particle!).