# Astrophysical Fluid Dynamics – Problem Set 6

Readings: Shu pages 73–81; pages 24–31 of Course Reader, photocopied from Frank, King, and Raine

**Problem 1.** Steady as She Blows

Consider spherically symmetric in/out-flow near an object of mass $M$. Assume the flow is isothermal at sound speed $c_s$. Define the Mach number $\mathcal{M} = u/c_s$ for radial velocity $u$.

(a) Show that

$$\left(\frac{1}{\mathcal{M}} - \mathcal{M}\right)\frac{\partial \mathcal{M}}{\partial r} = \frac{GM}{c_s^2 r^2} - \frac{2}{r} \tag{1}$$

where $r$ measures radial distance from the object and $G$ is the gravitational constant.

(b) Show that

$$\frac{\mathcal{M}^2}{2} - \ln \mathcal{M} = 2\frac{r_s}{r} + 2\ln\frac{r}{r_s} + C \tag{2}$$

where $r_s \equiv GM/2c_s^2$ gives the radius of the "sonic point" and $C$ is a constant of integration.

Plot several solutions of $\mathcal{M}$ versus $r$, drawing from all mathematically possible families. Label the solutions by their respective $C$ values. Identify those subset of solutions that are physically realizable.

**Problem 2.** The Solar Wind

At the base of the solar corona at $r \approx R_\odot$, outward flow speeds are still subsonic. Spectral line observations of the coronal base reveal temperatures of about $T \approx 2 \times 10^6$ K, a fully ionized plasma, and a number density of protons of about $10^8\ \mathrm{cm}^{-3}$.

(a) Estimate $\dot{M}$ for the solar wind, assuming the wind is isothermal. Express in $M_\odot\,\mathrm{yr}^{-1}$.

(b) Estimate the number density of solar wind protons flying past the Earth. Express in $\mathrm{cm}^{-3}$.

**Problem 3.** Jet Collimation by Pressure Confinement from an External Medium

Adapted from Pringle and King, problem 3.4. The problem starts by considering a de Laval nozzle, and ends by considering how astrophysical *jets* might behave like such nozzles—with the tube of given cross-sectional area replaced by an external medium of a given pressure profile.

Consider a $\gamma$-law gas moving inviscidly through a de Laval nozzle. The gas initially is very subsonic, but eventually moves supersonically. Take the sound speed of the gas initially to be $c_{s0}$.

(a) Solve for the gas velocity $u$ and the sound speed $c_s$ at the pinched throat of the nozzle, in terms of $c_{s0}$ and $\gamma$.

(b) Solve for the gas velocity $u$ at infinity, in terms of $c_{s0}$ and $\gamma$.

(c) Solve for the pressure $P$ at the pinched throat, in terms of the initial $P_0$ and $\gamma$. Does the pressure decrease monotonically everywhere along the tube? Increase monotonically? Neither? Justify your answer.

(d) Now consider an astrophysical jet. How are jets collimated? One proposed mechanism is pressure confinement by an external, static medium through which the jet travels. For example, the jet could be a stellar jet (powered by accretion from a disk), and the external medium could be the remnant gas cloud from which the star formed. Or the jet could be an AGN (supermassive black hole) jet (powered by accretion from a disk), and the external medium could be gas in the core of the galaxy.

There are two ideas here. The first is that the external, static medium has a thermal pressure that decreases monotonically in the jet direction. The second is that the jet and the external medium reach pressure equilibrium. That is, the thermal pressure at any point within the jet equals the local thermal pressure of the external surrounding medium. Once in pressure equilibrium, the jet stops expanding in the transverse direction (some papers refer to the jet as being "cocooned.") These two ingredients, coupled with the usual assumption of steady inviscid flow, imply a very close analogy between de Laval nozzles and jets. Instead of being given the tube's cross-sectional area $A(x)$ and solving for (say) $P(x)$, we are instead given $P(x)$ and solve instead for the jet cross-sectional area $A(x)$.

Consider a steady jet composed of a gas having $\gamma = 7/5$, travelling through an external medium whose pressure scales with distance as $x^{-2}$ (like that of a singular isothermal sphere). Neglect gravity. Very far along the jet, how does the *opening angle* of the jet (transverse length divided by longitudinal length $x$) scale with $x$? Take the jet to have a circular cross-section. Justify whatever assumptions you make.

(The idea of jet as de Laval nozzle was introduced by Blandford and Rees 1974, in the context of relativistic AGN jets. There also exists a Bernoulli constant for relativistic flows, but it looks different from our non-relativistic form $u^2/2 + \int dP/\rho + \Phi$. After thinking about this problem, you should find at least the first few pages of their paper to be fairly readable; you will also appreciate their Figure 1.)

**OPTIONAL Problem 4.** Isolated Accreting Compact Objects

(a) Estimate the accretion rate $\dot{M}$ of an isolated neutron star of mass $M = M_\odot$ travelling through the interstellar medium (ISM) at a velocity $v = 200\,\mathrm{km\,s^{-1}}$. Take the (warm component of the) ISM to have density $n = 1\,\mathrm{cm^{-3}}$ and temperature $T = 10^4\,\mathrm{K}$. Express in $M_\odot\,\mathrm{yr}^{-1}$.

(b) Estimate the accretion rate $\dot{M}$, in $M_\odot\,\mathrm{yr}^{-1}$, near the Bondi radius[1] of Sgr A*, the black hole at the center of our Galaxy having mass $M = 3 \times 10^6 M_\odot$. Chandra observations reveal that within 0.1 pc of Sgr A*, fully ionized plasma has density $n \approx 30\,\mathrm{cm^{-3}}$ and $kT_e \approx 1.3$ keV (Baganoff et al. 2003).[2] Neglect any relative bulk velocity between the plasma and Sgr A*.

[1]We need to add the qualifier "near the Bondi radius" because only a small fraction of the mass that flows past the Bondi radius actually makes it onto the black hole (past the Schwarzchild radius), given how the observed luminosity of Sgr A* is so low. Calculating this small fraction from first principles remains an unsolved problem.

[2]This plasma seems to originate from winds from nearby O stars.