Astrophysical Fluid Dynamics – Problem Set 8

Readings: Selections from Sturrock in the Course Reader

Problem 1. Rigidly Rotating Magnetospheres

A magnetized star is set spinning. The immediate environoment of the star—its *magnetosphere*, where the magnetic field of the star dominates all other sources of energy density—is filled with plasma. This problem considers how magnetospheric plasma moves.

Work in the inertial (lab) frame in which the star is spinning, and in cylindrical coordinates (d, ϕ, z), where d measures distance from the rotation axis of the star.

(a) Take the plasma to have effectively infinite electrical conductivity. Write down an expression for the electric field $\vec{E}$ in terms of the velocity of the plasma $\vec{v}$ and the magnetic field $\vec{B}$.

(b) Take the magnetosphere to be in steady state. Use Faraday's law of induction to prove that $E_\phi = 0$.

(c) Combine (a) and (b) to deduce that the plasma's poloidal (non-$\hat{\phi}$) velocity is parallel to the poloidal magnetic field.

(d) Use the induction equation for $\partial\vec{B}/\partial t$, the absence of magnetic monopoles, and the assumption that the flow is incompressible to find that

$$\frac{\partial\vec{B}}{\partial t} = (\vec{B}\cdot\nabla)\vec{v} - (\vec{v}\cdot\nabla)\vec{B} \tag{1}$$

(e) Assume the plasma's poloidal velocity is zero (as would be the case if the plasma is in hydrostatic equilibrium along a poloidal field line). That is, take $\vec{v}$ to be purely azimuthal:

$$\vec{v} = v(\vec{r})\hat{\phi} = \vec{\Omega}(\vec{r}) \times \vec{r} = [\Omega(\vec{r})\hat{z}] \times \vec{r} \tag{2}$$

where $\vec{r}$ is the displacement vector from the origin, and $\Omega\hat{z}$ is the local angular velocity of plasma. (At this stage, there is no reason to believe that Ω cannot vary arbitrarily with position, i.e., the plasma might be differentially rotating in an arbitrary way. In fact, though, part (f) will show that Ω is not free to vary arbitrarily.)

Prove from (1), the condition of steady state, and the absence of magnetic monopoles that

$$\frac{\partial B_d}{\partial\phi} = \frac{\partial B_\phi}{\partial\phi} = \frac{\partial B_z}{\partial\phi} = 0 \tag{3}$$

(Note: this is not the same as $\partial\vec{B}/\partial\phi = 0$, since unit vectors in cylindrical coordinates have non-zero phi-derivatives.)

(f) Show that (e) implies $(\vec{B}\cdot\nabla)\vec{\Omega} = 0$. That is, the angular velocity of plasma is constant along a field line. This result is Ferraro's (1937) Law of Iso-Rotation.

Every field line is rooted on the star. If the star is rigidly rotating, then Ferraro's Law implies that magnetospheric plasma is also rigidly rotating at the same angular velocity of the star. The plasma is definitely not moving on Kepler orbits.

Problem 2. Drift

This problem affords another perspective on Ferraro's Law.

Consider uniform electric $\vec{E}$ and magnetic $\vec{B}$ fields each oriented perpendicular to the other.

(a) Show that the motion of a charged particle can be decomposed into a fast gyromotion about $\vec{B}$ plus a slow *drift* at velocity

$$\vec{v}_D = c\frac{\vec{E}\times\vec{B}}{B^2} \tag{4}$$

This is a standard result that can be found in many textbooks. Please provide a complete derivation and also sketch the motions of both an electron and a proton, annotating your sketch with directions and magnitudes of motion.

(b) Verify the validity of (4) in the context of problem 1.

Problem 3. In the Beginning, the Biermann Battery

In class, in deriving Ohm's Law (the "difference" equation between the ion and electron momentum equations), we dropped a term that looked like

$$\frac{\partial}{\partial x_s}\left(-en_e\langle w_{er}w_{es}\rangle\right) \tag{5}$$

Restore this term, but keep only the diagonal components of the electron stress tensor, i.e., set $n_e m_e\langle w_{er}w_{es}\rangle = P_e\delta_{rs}$, where m_e is the mass of the electron, P_e is the electron partial pressure, and δ_{rs} is the Kronecker delta. (In other words, ignore the viscous contribution to the stress tensor and keep only the pressure contribution).

Derive the slightly more general form of the induction equation

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B}) + \frac{c^2}{4\pi\sigma}\nabla^2 B + \frac{c}{n_e^2 e}\nabla P_e \times \nabla n_e \tag{6}$$

The last term on the right-hand-side is the term introduced by Ludwig Biermann to generate magnetic fields from material that is initially field-free. It relies upon the contours of constant electron pressure being misaligned with the contours of constant electron density to pressure-accelerate electrons away from ions in such a way as to produce current (which in turn produces field). The action is similar to that of a battery (which relies on chemical reactions to accelerate electrons away from ions), and for this reason we call this term the Biermann battery term. It has been invoked to generate truly primordial magnetic field in proto-galactic plasma (see Kulsrud's (1997) article in "Critical Dialogues in Cosmology.")