Astro 162 – Planetary Astrophysics – Solutions to Set 10

Problem 1. *Dogs and Cats Living Together*

Estimate the size of the meteorite that crashed into the Earth 65 million years ago to form the Chicxulub crater and to wipe out life as the dinosaurs knew it. We saw in class that the crater was about 180 km in diameter.

The crater is larger than the critical size, ∼7 km, above which it takes more energy to lift material out of the crater against gravity than it does to shatter the material. So we are in the gravity regime. In the gravity regime, the crater diameter $D \sim (2v^2/g)^{1/4} d^{3/4}$, where v is the impact velocity, $g = 10^3$ [cgs] is the gravitational acceleration at the Earth's surface, and d is the projectile diameter. Take $D = 180\,\mathrm{km}$ and $v \approx 45\,\mathrm{km/s}$ (between 30 km/s = orbital velocity of Earth and 60 km/s = relative velocity between Earth and a body revolving about the Sun in a retrograde orbit at 1 AU) and solve for $\boxed{d \sim 12\,\mathrm{km}}$.

Note that this rock is so large that it will fail to be slowed by the atmosphere of the Earth, justifying our assumption of neglecting air drag for our estimate of v. The critical size above which solid, competent spheres fail to be slowed is about ∼1 meter, as we showed in a previous problem set. (The Space Shuttle is larger than ∼1 meter and it *does* get slowed down by air, but it is not a solid, competent sphere: it is a plane, whose area-to-mass ratio is much greater than that of a sphere).

Problem 2. J_2

A body's J_2 has simple physical interpretations. Its order of magnitude can be estimated as either

$$J_2 \sim \frac{R_{\mathrm{equator}} - R_{\mathrm{pole}}}{R_{\mathrm{equator}}} \tag{1}$$

where R_{equator} is the body's equatorial radius (measured perpendicular to the spin axis) and R_{pole} is the body's polar radius (measured parallel to its spin axis). Alternatively, we can estimate it as

$$J_2 \sim \frac{\text{Rotational Kinetic Energy}}{\text{Gravitational Potential Energy}}\,. \tag{2}$$

(a) Use (2) to derive an analytic expression for J_2 in terms of the spin frequency of the planet (Ω), the mean density of the planet (ρ), and any natural constants.

The numerator equals $I\Omega^2/2$, where $I = 2MR^2/5$ for a uniform density sphere. The denominator equals $3GM^2/R$ for a uniform density sphere (derived by assembling the planet shell-by-shell, as you have done in freshman mechanics). Putting top over bottom, we get

$$J_2 \sim \frac{R^3\Omega^2}{3GM} \tag{3}$$

$$\sim \boxed{\frac{\Omega^2}{4\pi G\rho}} \tag{4}$$

(b) Estimate J_2 for the Earth and for Saturn without looking up the answers directly. Then compare your answers to the truth by looking up the answers directly.

The mean density of Earth is $\rho = 5.5\,\mathrm{g\,cm^{-3}}$. Its rotation frequency is $\Omega = 2\pi/(24\,\mathrm{hr})$. Then for Earth, $\boxed{J_{2,E} \sim 0.0011}$. The truth is $\boxed{J_{2,E} = 0.001083}$, nearly exactly right.

The mean density of Saturn is $\rho = 0.7\,\mathrm{g\,cm^{-3}}$. Its rotation frequency is $\Omega = 2\pi/(10.8\,\mathrm{hr})$. Then for Saturn, $\boxed{J_{2,S} \sim 0.04}$. The truth is $\boxed{J_{2,S} = 0.016298}$, a factor of 2 smaller than our estimate (still big enough that you can tell just by looking at Saturn that it is not a sphere). Our error is larger for Saturn than for Earth because the uniform density approximation is less good for Saturn (a highly compressible gas/liquid) than for Earth.

Problem 3. *Weirdness on the Sphere*

a) Long-period comets, unlike their short-period brethren, come from all directions on the sky. We say that their orbital planes are distributed isotropically; i.e., the orbit pole vector (the vector perpendicular to the plane of a comet's orbit) can point anywhere on the celestial sphere with equal probability.

Define the orbital inclination, i, to be the angle between the orbital plane of a comet and some fixed reference plane. The inclination can take any value between $i = 0$ and $i = 180°$. "Prograde" orbits have $0 \le i < 90°$, "retrograde" orbits have $90° < i \le 180°$, and "polar" orbits have $i = 90°$.

Derive the differential inclination distribution, dN/di, for isotropically oriented cometary orbits, where dN is the differential number of comets having inclinations between i and $i + di$. Normalize your distribution so that $\int_0^{180°} (dN/di)di = 1$.

First note that co-planar orbits are LESS common than polar orbits. There is only 1 orbit having $i = 0$: the orbit whose pole vector points straight up, perpendicular to the reference plane. By contrast, the pole vectors of polar orbits can point anywhere along the circumference of the celestial equator.

The differential number of orbits having i between i and $i + di$ is proportional to the differential swath of solid angle (on the celestial sphere centered on the Sun) swept out by pole vectors inclined between i and $i + di$ with respect to the reference plane and having arbitary azimuthal orientation. This differential solid angle equals $2\pi \sin i \, di$. Then $dN \propto 2\pi \sin i di$, which means $dN/di \propto \sin i$ (where we have dropped the 2π since it is a constant). Normalizing this distribution gives $\boxed{dN/di = (\sin i)/2}$.

(b) Given your answer in (a), are nearly co-planar ($i \approx 0$) orbits more common, less common, or just as common to find as nearly polar ($i \approx 90°$) orbits? (Are you weirded out by your answer?)

We already answered this question in (a). Since $dN/di \propto \sin i$ INCREASES with i, co-planar orbits are $\boxed{\text{LESS COMMON}}$ than polar orbits. Initially I was weirded out by this result, but now I find it a natural consequence of phase space, spherical geometry, and the breaking of symmetry due to the introduction of an arbitrary reference plane.

(c) Geoff Marcy and his collaborators measure the line-of-sight Doppler shifts of stars to infer the presence of planetary companions around those stars. The extrasolar planet tugs gravitationally on its parent star and induces the star to revolve about the common center of mass. A key unknown in the measurement is the angle between the orbital plane of the planet and the plane of the sky (the plane of the sky is perpendicular to the observer's line of sight). Call this angle i. If $i = 0$, the orbit plane coincides with the sky plane and there will be no line-of-sight Doppler shift measurable. The maximum line-of-sight Doppler shift arises for $i = 90°$. Here, $0 \leq i \leq 90°$ only.

For a given measured Doppler shift, Marcy et al. cannot measure the mass of the planet, m, directly. They can only measure $m \sin i$ because of the unknown orientation of the planet's orbital plane. For example, if Marcy et al. measure an $m \sin i = 1$ Jupiter mass, they don't know whether it's a 1 Jupiter mass planet whose orbit is oriented perpendicular to the sky plane ($i = 90°$), or whether it's a 2 Jupiter mass planet for which $i = 30°$. (And in the ridiculously extreme limit, they don't even know if it's an infinite mass planet with $i = 0°$!)

Fischer et al. measured the Doppler wobble of the star HD217107 and derive $m \sin i = 1.23 M_J$, where M_J is a Jupiter mass. Calculate the probability that the unseen companion is not a planet but a star whose mass exceeds 80 Jupiter masses.

If $|\sin i| < 1.23 M_J / 80 M_J$, then the companion mass exceeds $80 M_J$ and is a star. The fraction of orbits for which this is true is just given by the relevant integral over the distribution we derived in (a): $2 \times \int_0^{\arcsin(1.23/80)} [(\sin i)/2] di = \boxed{1.2 \times 10^{-4}}$, where we have remembered to multiply by 2 because $i \approx 180°$ also gives large masses just like $i \approx 0°$.

Problem 4. *The Nougat in the Center: Isothermal Extreme*

Repeat Problem 4 of last week's problem set #9, but assume instead that the gas behaves *isothermally when it accretes onto the rocky core, not adiabatically as was assumed last week. That is, assume this time that $T_s = T_0$.*

(a) Find a new analytic expression for ρ_s, using the same procedure as described in last week's problem set.

Hydrostatic equilibrium for the atmosphere reads:

$$\frac{1}{\rho}\frac{dP}{dr} = -g \tag{5}$$

For an ideal, isothermal gas, the pressure $P = \rho c_s^2$, where $c_s^2 = kT_0/\mu$. The gravitational acceleration $g = GM/r^2$. Plugging in, we have

$$c_s^2 \frac{d\ln\rho}{dr} = -\frac{GM}{r^2} \tag{6}$$

Integrate this equation to find:

$$\ln\rho = \frac{GM}{c_s^2}\frac{1}{r} + C \tag{7}$$

where C is the constant of integration. Now use the boundary condition: at $r \to \infty$, $\rho = \rho_0$. Therefore

$$\rho = \rho_0 \exp(GM/rc_s^2) \tag{8}$$

The atmospheric density at the surface of the core is $\boxed{\rho_s = \rho_0 \exp(GM/Rc_s^2)}$.

(b) Solve for a new critical radius of the core, $R_{\rm c}$, such that $M_{\rm env} = M$. Give an analytic expression for $R_{\rm c}$ in terms of the variables given. Also give a numerical estimate for the critical mass $M_{\rm c}$ in terms of $M_\oplus$.

For actual numbers, see last week's problem set.

In general, isothermal and adiabatic conditions represent two extremes. The isothermal condition assumes that the gas is able to cool perfectly (say, by radiation to space) back down to its original temperature even when it undergoes compression as it accretes onto the core. The adiabatic condition says that the gas is totally unable to cool and therefore heats up (as if it were wrapped in insulating styrofoam) when it undergoes compression as it accretes onto the core. The truth lies somewhere in between the isothermal assumption *and the adiabatic assumption. The true critical core mass (5–10 $M_\oplus$) should lie in between the answer you get this week and last week's answer.*

Following the procedure from last week's set, the mass of the atmosphere is

$$M_{\rm env} \sim 4\pi R^2 H \rho_s \tag{9}$$

$$M_{\rm env} \sim 4\pi R^2 \frac{R^2 c_s^2}{GM} \rho_0 \exp(GM/Rc_s^2)\,. \tag{10}$$

Set $M_{\rm env} = M$ and re-write the last equation above as

$$\frac{4\pi R^4 c_s^2 \rho_0}{GM^2} \exp(GM/Rc_s^2) = 1 \tag{11}$$

Now $M = 4\pi\rho_M R^3/3$, where ρ_M is the mass density of the core. Substitute this relation into the previous one, and define $x = GM/Rc_s^2$ to write

$$\frac{3}{\rho_M}\frac{\rho_0}{x}\exp(x) = 1 \tag{12}$$

This is an equation for x. Take $\rho_M \approx 3\,{\rm g\,cm^{-3}}$ and $\rho_0 \approx 10^{-11}\,{\rm g\,cm^{-3}}$. Then trial-and-error on my pocket calculator gives

$$x \approx 29 \quad = \quad \frac{GM}{Rc_s^2} \tag{13}$$

$$= \quad \frac{4\pi G\rho_M R^2}{3c_s^2} \tag{14}$$

Take $c_s \approx \sqrt{kT_0/\mu} \sim 6\times10^4\,{\rm cm\,s^{-1}}$ and solve for $R = 3456\,{\rm km}$.
Then $\boxed{M \sim 5\times10^{26}\,{\rm g} \sim 0.09 M_\oplus}$. This is less than what people find in numerical simulations (5–10 $M_\oplus$), as we would expect it to be, since the truth lies between the isothermal and adiabatic extremes. Thus, between last week's problem and this week's problem, we have bracketed the truth.

Curiously, the transcendental equation (12) actually has another solution $x \ll 1$. This gives a critical core mass which is much, much smaller than the one we have derived.