Astro 162 – Planetary Astrophysics – Solutions to Set 11

Problem 1. *Tearing Apart the Oort Cloud*

(a) Consider an invading star of mass $M_* = 0.5M_\odot$ speeding through the Oort Cloud at a velocity relative to the Sun of $V_* = 30\,\mathrm{km/s}$. Idealize the Oort Cloud as a shell of comets at heliocentric distance $R = 10^5\,\mathrm{AU}$. Estimate how close the invading star needs to come near a comet to gravitationally unbind that comet from the Oort Cloud. Call this maximum approach distance d (so for all impact parameters less than d, the comet is likely to get thrown out of the Cloud). You may use the impulse approximation as described in class, but only if you justify its use.

When the invader star comes within distance d of the comet, the gravitational acceleration which the comet feels is GM_*/d^2. The comet feels this acceleration for a time of order $2d/V_*$. Then after the encounter, the comet has picked up an additional velocity $\Delta V \sim 2GM_*/dV_*$. To unbind the comet from the star, this additional velocity must be of order the Keplerian orbital velocity at the comet's location, $V_K = \sqrt{GM_\odot/R}$. Set $\Delta V = V_K$ and solve for $\boxed{d \sim 2GM_*\sqrt{R}/V_*\sqrt{GM_\odot} \sim 300\,\mathrm{AU}}$.

Note that we have made two approximations. One is the impulse approximation for our calculation of ΔV. This is justified because the duration of the encounter, $2d/V_*$, is much smaller than the time over which the comet changes its position relative to the Sun, which is just the Kepler orbital period. The second approximation lies in our neglect of the perturbation to the Sun by the invader star. This is also justified because $d \ll R$.

(b) Over the age of the solar system ($4 \times 10^9\,\mathrm{yr}$), what fraction of Oort Cloud comets at $R = 10^5\,\mathrm{AU}$ are likely to be ejected into interstellar space by such single, violent scatterings? Take the mean stellar number density to be 0.1 stars/pc^3, the mean stellar mass to be M_* as above, and the stellar velocity dispersion to be V_* as above.

Each close stellar passage like the one we are considering drills a "cylinder of destruction" through the Oort Cloud. For a shell of comets of heliocentric radius R, two circular patches, each of area πd^2, are drilled out of the shell per invader star that comes within R of the Sun. How many invader stars pierce the shell over the age of the solar system? Answer: $n_*\pi R^2 V_* t$, where n_* is the number density of stars in space and t is the age of the solar system. Then the total drilled area over the age of the solar system is $n_*\pi R^2 V_* t(2\pi d^2)$. Divide this by the total surface area of the comet shell, $4\pi R^2$, to get the fraction of comets that have been dislodged: $f = \pi n_* V_* t d^2/2 = 2\pi n_* t G M_*^2 R/V_* M_\odot$, where for the last equality we have inserted our expression for d from (a). Plug in numbers to find the fraction $\boxed{f \approx 5\%}$.

(c) Repeat (b), but for $R = 10^4\,\mathrm{AU}$.

Note that in our last symbolic expression in (b), $\boxed{f \propto R}$. So reducing R by a factor of 10 reduces f by a factor of 10 to $\boxed{0.5\%}$. Some of you blithely assumed that since the encounter rate scales as the cross-section R^2, that $f \propto R^2$, but this neglects the R-dependence of d! Moral: reduce all symbolic expressions into irreducible variables before you deduce scaling relations!

Problem 2. *Bye-Bye Bopp, So Long SOHO*

(a) Take Comet Hale-Bopp to be a sphere of dirtyish water ice of radius $R \sim 10\,\mathrm{km}$ *and albedo* $A \sim 0.1$. *Water ice that is naked before the Sun tends to sublimate at heliocentric distances inside about 3 AU.*

Hale-Bopp's perihelion distance is $q = 0.91\,\mathrm{AU}$ *and its aphelion distance is* $Q = 372\,\mathrm{AU}$. *ESTIMATE the number of times, N, that Hale-Bopp will loop around the sun before it evaporates, assuming the comet keeps the same orbit it has today.*

When the comet is at heliocentric distances r between 3 AU and $q = 0.91$ AU, let's take all of the absorbed sunlight to go into sublimation. This is pretty much of all of the incident sunlight, since $1 - A \approx 1$. The amount of energy the comet absorbs per time is $(L_\odot/4\pi r^2)\pi R^2$. Convert this into a mass loss rate per time by dividing by the latent heat of vaporization of water, $L_{vap} = 2\times 10^{10}\,\mathrm{erg\,g^{-1}}$. Convert this further into a rate of *radius decrease* per time by dividing by the shell surface area, $4\pi R^2$, and the internal density of the comet, ρ. Then over a heating time interval, P, the comet decreases in radius by $\Delta R = (L_\odot/4\pi r^2)(P/4L_{vap}\rho)$. What is P? It is of order the Keplerian orbital period at these distances, $P \sim r^{3/2}/\sqrt{GM_\odot}$. Then $\Delta R = (L_\odot/r^{1/2}16\pi\rho L_{vap}\sqrt{GM_\odot})$. The comet survives $N = R/\Delta R$ passes. Plugging in numbers like $\rho = 1\,\mathrm{g\,cm^{-3}}$ and a characteristic $r = 1$ AU (note that r is in the denominator in ΔR, so most the sublimation occurs at small r), I get $\boxed{N \sim 10^4}$ passes before the comet is gone. The rule of thumb to remember is that each perihelion passage near $r = 1\,\mathrm{AU}$ wicks off about 1 meter's worth of ice off the the surface of the comet, independent of the size of the comet for $R > 1\,\mathrm{m}$.

(b) SOHO (SOlar and Heliospheric Observatory) is a satellite that stares at the Sun 24 hours a day, relaying ultraviolet images of the Sun back to Earth every 30 minutes. SOHO lives at the second Lagrange point (L2) between the Earth and the Sun (yes, we said in class that this point was dynamically unstable and it is, but the effective potential is so flat near this saddle point that periodically firing tiny thrusters onboard SOHO can keep the spacecraft near L2 and pointing at the Sun at all times.)

An unexpected side benefit of SOHO was that it discovered thousands of sun-grazing comets. These comets had perihelia ranging from $q \sim 1$–10 *solar radii. In fact, SOHO is history's most successful comet-hunter, having discovered 1200+ comets in 5+ years. An image of a sun-grazer can be found at*
http://umbra.nascom.nasa.gov/comets/twin_comets_19980601.gif

SOHO sees these comets going towards the sun, but it doesn't see them coming away.

Given this observation, estimate the maximum radius that these sun-grazers could have.

SOHO doesn't see the comet post-perihelion because the comet has evaporated away. Set $R_{max} = \Delta R$, where ΔR is the expression in (a) for the radius decrease after a single perihelion passage. The only parameter which has changed between (a) and (b) is r; now the characteristic distance is $r \approx 3R_{\odot} \approx 0.01\,\mathrm{AU}$. We scale our answer in (a) for ΔR by $(0.01)^{-1/2} = 10$ to get $\boxed{R_{max} = \Delta R \sim 10\,\mathrm{m}}$. Some of you were surprised by the smallness of this value, but you shouldn't be; in fact, most sun-grazers belong to the Kreutz family of comets, fragments of a once-larger parent body.

Problem 3. *Bubble Bubble, Toil and Trouble*

The pressure inside a bubble differs slightly from the pressure outside. Use the Buckingham Pi Theorem (otherwise known as dimensional analysis) to write down an analytic expression for this pressure difference.

Salient properties of liquid bubbles:

- *The surface consists of a liquid.*
- *The interior contains a gas.*
- *The bubble may be embedded in a gas or liquid.*

Examples:

- *Soap bubble of air in air (surface is made of liquid soap).*
- *Champagne bubbles (surface is made of liquid champagne; interior is largely carbon dioxide).*
- *Coke bubbles (surface is made of liquid Coke; interior is largely carbon dioxide).*
- *Bubbles exhaled from scuba diver (surface is made of water; interior is largely carbon dioxide and nitrogen).*

Ask whether the pressure inside the bubble is greater than or less than the pressure outside.

You answer should suggest why it is that tiny bubbles in Coke cans make a huge racket.

This turned out to be a bit of a trick question.

The physical variables include at least: (1) the pressure difference, ΔP; and (2) the bubble radius, R. We use our amazing physical intuition to deduce that another physical variable that must be relevant is (3) the surface tension of the bubble wall, γ. The pressure inside the bubble is greater than the bubble outside, which makes the bubble want to burst. It is the surface tension of the bubble wall which holds the bubble wall molecules together and prevents the bubble from bursting.

Therefore, the number of physical variables is $m = 3$. We might be tempted to choose a fourth variable like $\Delta\rho$, the density difference between the outside and the inside. The main reason why we should not choose $\Delta\rho$ is that from the perspective of FORCE balance, $\Delta\rho$ is not important. The reason why there is a force trying to burst the bubble is because there is a pressure difference, ΔP (after all, force is a PRESSURE times an area). The $\Delta\rho$ (and the ΔT temperature difference between the outside and the inside) is subsumed by ΔP.

The number of INDEPENDENT fundamental quantities LOOKS LIKE it's $n = 3$. The pressure difference ΔP has units of $[M]/([T]^2[L])$. The surface tension γ has units of energy/area, or $[M]/[T]^2$. And the bubble radius has units of length, $[L]$. Therefore we might conclude that the number of independent fundamental quantities is 3 (mass $[M]$, length $[L]$, and time $[T]$). Unfortunately, that is incorrect because in ΔP and γ, MASS and TIME always enter as the SAME COMBINATION: namely, $[M]/[T]^2$. Therefore mass and time cannot be counted as separate independent quantities; only the combination $[M]/[T]^2$ can be counted as 1 independent fundamental quantity. Length $[L]$ remains the another independent fundamental quantity. Therefore the number of independent fundamental quantities is actually $n = 2$.

The Buckingham Pi Theorem states that we can form $m - n$ dimensionless product groups. For this problem, $m - n = 1$. Therefore we can form only 1 group. The choice is necessarily unique:

$$\frac{R\Delta P}{\gamma} = \Pi_1 \tag{1}$$

where Π_1 is some dimensionless number that we cannot know unless we make experimental measurements or perform a detailed calculation. Therefore

$$\boxed{\Delta P = \frac{\Pi_1 \gamma}{R}}, \tag{2}$$

is the final answer. A proper calculation gives $\Pi_1 = 4$ for a soap bubble of air in air, and $\Pi_1 = 2$ for a bubble of carbon dioxide in Coke.

The pressure difference in small bubbles is bigger than for large bubbles ($\Delta P \propto 1/R$). Indeed, the bursting of tiny bubbles in Coke can release gas of very high pressure and thereby generate large-amplitude sound waves (i.e., loud noise).