Astro 162 – Planetary Astrophysics – Solutions to Set 12

Problem 1. *Soup Bowls and Ocean Basins*

A flat bottomed bowl of radius R is filled with water to depth H.

(a) Derive an approximate analytic formula for the period P of the sloshing mode.

As described in lecture, the sloshing mode is a shallow water wave. That is, the entire layer of water participates in the mode, not just the surface. We know that for shallow water waves, the dispersion relation reads

$$\omega = k\sqrt{gH} \tag{1}$$

where $k = 2\pi/\lambda$ is the wavenumber. Now from crest to trough is $\lambda/2$. When all the water is heaped up on one side of the bowl, the distance from crest to trough is $2R = \lambda/2$. Therefore $\lambda = 4R$ and $k = \pi/(2R)$. Since $\omega = 2\pi/P$ where P is the wave period, we find that

$$\boxed{P = \frac{4R}{\sqrt{gH}}}. \tag{2}$$

(b) Use your formula to estimate P for a soup bowl and for the Earth's ocean basin.

For a soup bowl, $R \sim 7\,\mathrm{cm}$ and $H \sim 5\,\mathrm{cm}$. Then $\boxed{P_{\mathrm{soup}} \sim 0.4\,\mathrm{s}}$.

For the Earth's ocean basin, $R \sim 6000\,\mathrm{km}$ and $H \sim 5\,\mathrm{km}$. Then $\boxed{P_{\mathrm{ocean}} \sim 30\,\mathrm{hr}}$. The fact that this ocean-sloshing period is close to the synodic period between the Earth's rotation and the Moon's revolution about the Earth means that the Moon can resonantly excite tides to large amplitude.

Problem 2. *The Storm That Launched a Thousand Waves*

Following a winter storm above the ocean, the interval between waves at California beaches declined from 17–19 s on Sunday, to 16–18s on Monday, and to 15–16s on Tuesday. Typical values are 10–11 s.

These waves are surface gravity waves on deep water (or somewhat misleadingly, "deep water waves.")

(a) What was the maximum sustained wind speed during the storm?

For surface gravity waves on deep water, the dispersion relation reads

$$\omega = \sqrt{gk}\,. \tag{3}$$

The phase speed of such waves is

$$v_{\rm ph} = \omega/k = \sqrt{\frac{g}{k}} \tag{4}$$

while the group speed is

$$v_{\rm gp} = d\omega/dk = \frac{1}{2}\sqrt{\frac{g}{k}}\,. \tag{5}$$

From lecture, we know that the wind blowing across the surface of the water generates waves. Moreover, from lecture, the wind speed equals the PHASE speed of the waves generated. So to determine the maximum wind speed during the storm, we need to determine the maximum phase speed of the waves generated. Of the waves that we observe at the beach, which ones have the maximum phase speed? By equation (4), the phase speed goes as $1/\sqrt{k}$. So we need to identify the waves having the smallest k to maximize $v_{\rm ph}$.

What we measure is ω, the frequency of waves crashing on the beach. From (3),

$$k = \frac{\omega^2}{g}\,. \tag{6}$$

Plug this into (4):

$$v_{\rm ph} = \frac{g}{\omega} = \frac{gP}{2\pi} \tag{7}$$

where $\omega = 2\pi/P$ and P is the wave period. So the longest period wave has the greatest phase speed. The longest period we observed at the beach is 19 seconds. Plug in $P = 19\,{\rm s}$ and $g = 10^3$ [cgs] to find

$$\boxed{\max(v_{\rm wind}) = \max(v_{\rm ph}) = 3000\,{\rm cm\,s^{-1}} = 30\,{\rm m\,s^{-1}} = 100\,{\rm kph}}\,. \tag{8}$$

(b) How distant was the storm from the beaches?

Surface gravity waves are dispersive; in other words, waves of different frequencies travel at different speeds. That is why the longest period waves arrive first.

Call the distance between the storm and us D. It took Sunday's waves a time Δt_{Sunday} to travel to us:

$$\Delta t_{\text{Sunday}} = \frac{D}{v_{\text{gp},18}} \tag{9}$$

Notice the GROUP velocity enters here: the entire group of waves travels at the group velocity from the storm to us. Also note the subscript "18," which is the average wave period (in seconds) for Sunday.

It took Tuesday's waves a time $\Delta t_{\text{Tuesday}} = \Delta t_{\text{Sunday}} + 2$ days to reach us:

$$\Delta t_{\text{Tuesday}} = \Delta t_{\text{Sunday}} + 2\,\text{days} = \frac{D}{v_{\text{gp},15.5}} \tag{10}$$

where 15.5 seconds is the average wave period for Tuesday's waves. We can now use (9) and (10) to solve for the desired D. Subtract these equations to eliminate Δt_{Sunday}:

$$2\,\text{days} = \frac{D}{v_{\text{gp},15.5}} - \frac{D}{v_{\text{gp},18}} = D\left(\frac{1}{v_{\text{gp},15,5}} - \frac{1}{v_{\text{gp},18}}\right). \tag{11}$$

Combine (3) and (5) to find:

$$v_{\text{gp}} = \frac{1}{2}\frac{g}{\omega} = \frac{gP}{4\pi}. \tag{12}$$

Use $P = 18\,\text{s}$ and $P = 15.5\,\text{s}$ to solve for $v_{\text{gp},18}$ and $v_{\text{gp},15.5}$, respectively. Then use (11) to find that $\boxed{D \sim 15000\,\text{km}}$, or roughly halfway around the world! (The Pacific Ocean is a big place!)

Notice we have not used Monday's data explicitly, but have simply taken Sunday and Tuesday as endpoint data (two points make a line). You could have used Monday's data, too, and gotten estimates for D that agree to within a factor of 2.

(c) How long ago did the storm take place?

We can just use (9) and our answer to (b) to calculate $\boxed{\Delta t_{\text{Sunday}} \sim 12\,\text{days}}$. So the storm took place about $\boxed{\text{12 days prior to Sunday}}$.

(d) What are upper limits on the size and duration of the storm?

To estimate an upper limit on the (horizontal) size of the storm, assume first that the storm occurred instantaneously in time (that is, assume the duration of the storm is 0 seconds). We'll switch roles in this assumption later (that is, to estimate the upper limit on the duration of the storm, we'll assume the storm occurred at a single point in space.)

On Sunday, waves having a variety of periods, 17–19 seconds, crashed simultaneously on the beach. The 19 s waves travel faster than the 17 s waves [see equation (12)]. Let's say the 19 s waves were born a distance $D+\Delta D$ away. And let's say the 17 s waves were born a distance $D-\Delta D$ away. They arrived at the beach on the same day (Sunday). Therefore

$$\frac{D+\Delta D}{v_{\mathrm{gp},19}} = \frac{D-\Delta D}{v_{\mathrm{gp},17}} \tag{13}$$

We know every variable in this equation except for the desired ΔD (the radius of the storm). Straightforward algebra gives

$$\Delta D = D\left(\frac{v_{\mathrm{gp},19} - v_{\mathrm{gp},17}}{v_{\mathrm{gp},19} + v_{\mathrm{gp},17}}\right) = D\left(\frac{19-17}{19+17}\right) = \boxed{800\,\mathrm{km}}. \tag{14}$$

Now we estimate the upper limit on the duration of the storm. Assume the storm occurred at a single point in space. Again, what we know is that on Sunday, the 19 s waves and the 17 s waves arrived at the beach at the same time. Let's say that at the beginning of the storm, the wind speed was such that it generated only 17 s waves. And let's say that at the end of the storm, after time δt has elapsed, the wind speed was such that it generated only 19 s waves. Even though 17 s waves have a head-start in time, the 19 s waves travel faster than the 17 s waves and therefore both 17 and 19 s waves arrive at the beach on the same day (Sunday). Both sets of waves travel the same distance:

$$v_{\mathrm{gp},17}\Delta t_{\mathrm{Sunday}} = v_{\mathrm{gp},19}(\Delta t_{\mathrm{Sunday}} - \delta t) \tag{15}$$

which we can solve for

$$\Delta t = \Delta t_{\mathrm{Sunday}}\left(1 - \frac{v_{\mathrm{gp},17}}{v_{\mathrm{gp},19}}\right) = 12\,\mathrm{days}\left(1 - \frac{17}{19}\right) = \boxed{1\,\mathrm{day}}. \tag{16}$$

Now 1 day and 800 km (radius) are *upper* limits because the truth lies in between our dual assumptions of 0-duration/finite-size and 0-size/finite-duration.

Problem 3. *Tidal Disruption and the Roche Zone*

This problem examines why ring systems about all the giant planets occupy planetocentric distances that are less than ∼2 planetary radii.

a) Consider a perfectly rigid, spherical satellite of radius R_s, mass m_s, and density ρ_s orbiting a planet of radius R_p, mass m_p, and density ρ_p. Assume the satellite to be in synchronous lock; this means its spin period matches its orbital period. Take the satellite's orbital semi-major axis to be a_s and its orbital eccentricity to be zero.

A marble rests on the surface of this spinning satellite. The spin of the satellite tries to spin it off. The tidal field of the planet also tries to pull it off. The only force trying to keep it glued to the satellite is the satellite's own gravity. For small enough a_s, the marble will fly off. What is this minimum semi-major axis, $a_{s,1}$? Express in terms of ρ_s, ρ_p, and R_p.

First decide at which location on the satellite the marble is most unstable. Is it on the pole of the planet or on the equator? Is it right between the planet and the satellite? On the far side of the satellite away from the planet? At right angles with the center of the satellite and the planet? Once you decide the most unstable location, write down force balance for the marble at that location.

The marble is at its most unstable when it lies right in between the planet and the host satellite. Then the tidal acceleration from the planet acting to pull the marble off is $|(d/da)(Gm_p/a^2)R_s| = 2(Gm_p/a^3)R_s$. The centrifugal acceleration of the satellite acting to spin the marble off is $(Gm_p/a^3)R_s$. At $a = a_{s,1}$, these accelerations add to barely balance the satellite's gravitational pull, Gm_s/R_s^2. Then

$$\frac{2Gm_p}{a^3}R_s + \omega^2 R_s = \frac{Gm_s}{R_s^2} \quad (17)$$

$$3\frac{Gm_p}{a_{s,1}^3}R_s = \frac{Gm_s}{R_s^2} \quad (18)$$

Solve for

$$\boxed{a_{s,1} = \left(\frac{3\rho_p}{\rho_s}\right)^{1/3} R_p = 1.44\left(\frac{\rho_p}{\rho_s}\right)^{1/3} R_p} \quad (19)$$

b) Now consider a marble floating on a perfectly strengthless, fluid, synchronously rotating satellite. The satellite's shape is now free to distort because the satellite is sitting in the tidal field of the planet and because the satellite is spinning.

ESTIMATE the semi-major axis of the satellite, $a_{s,2}$, inside of which the marble flies off the watery satellite. This is a repeat of (a) except that you will need to account for *the distorted shape of the satellite; the satellite has an enhanced size due not only to the TIDE raised on it by the planet, but also due to its SPIN.*

You should at least decide whether $a_{s,2}$ should be larger or smaller than $a_{s,1}$.

Bring to bear all your powers of estimation. Credit will be awarded for precision and elegance.

The marble now floats an extra distance away from the center of the satellite. Let's estimate the new distance as $R' = R_s[1 + \Delta]$. Now force balance for the marble reads:

$$\frac{2Gm_p}{a^3} R_s(1+\Delta) + \omega^2 R_s(1+\Delta) = \frac{Gm_s}{R_s^2(1+\Delta)^2} + \frac{G\rho\pi R_s^2(\Delta \times R_s)}{R_s^2} \tag{20}$$

Every term is analogous to terms in part (a) except for the last term on the right-hand-side of the equation. This extra term arises because now there is extra mass beneath the marble. We have approximated this extra mass as a partial spherical shell of mass $\rho\pi R_s^2(\Delta \times R_s)$, and radius R_s.

For a synchronous satellite, the spin $\omega^2 = Gm_p/a^3$. Then equation (20) becomes

$$\frac{3Gm_p}{a^3}(1+\Delta) = \frac{Gm_s}{R_s^3(1+\Delta)^2} + \frac{Gm_s\Delta}{R_s^3} \tag{21}$$

where we have taken $m_s \approx \pi\rho R_s^3$ and divided through by R_s.

What is Δ? There are two contributions of comparable magnitude to the extra height. One is from the tide raised on the satellite by the planet, $\Delta_1 \approx (m_p/m_s)(R_s/a)^3$. The other is from the spin of the satellite, $\Delta_2 \approx \omega^2 R_s^3/Gm_s$. Since $\omega^2 = Gm_p/a^3$, we have $\Delta = \Delta_1 + \Delta_2 = 2(m_p/m_s)(R_s/a)^3$. With this relation, we can re-write (21) as

$$\frac{3}{2} = \frac{1}{\Delta(1+\Delta)^3} + \frac{1}{1+\Delta} \tag{22}$$

My pocket calculator gives $\Delta \approx 0.42$. Then

$$a_{s,2} = \left(\frac{2}{0.42}\right)^{1/3} \left(\frac{m_p}{m_s}\right)^{1/3} R_s \tag{23}$$

$$\boxed{a_{s,2} = 1.7 \left(\frac{\rho_p}{\rho_s}\right)^{1/3} R_p} \tag{24}$$

Chandrasekhar did a more proper job and got the coefficient to be 2.46 instead of our 1.7.

c) At orbital semi-major axes of less than $\sim$*2 planetary radii, there are no satellites whose sizes exceed 100 km, but there are rings composed of meter-sized boulders and smaller debris, and 10 km-sized satellites. Given your answers in (a) and (b), explain why these observations make sense.*

Since $\rho_p/\rho_s \approx 1$, we should expect bodies which are held together mostly by self-gravity to be ripped apart by *both* tidal and centrifugal forces if they are found at planetocentric distances less than $\sim 2R_p$. Small bodies which are held together mostly by intermolecular cohesive forces rather than self-gravity are immune to this disruption. Thus, plenty of small bodies—ring particles, small satellites—can exist within $\sim 2R_p$, but large bodies are torn apart. The tidal/centrifugal disruption zone is referred to as the "Roche zone."