

## Astro 162 – Planetary Astrophysics – Solutions to Problem Set 1

### Problem 1. Cooling Molecular Clouds and Star Formation

Consider again the gravitational collapse of a  $\sim 1 M_{\odot}$  clump of gas and dust of radius  $\sim 0.1$  pc. Assume the temperature of the cloud is such that the cloud is marginally Jeans-unstable. The gas consists predominantly of molecular hydrogen.

As the cloud collapses inward, its density increases. In the absence of cooling mechanisms, the gas will heat adiabatically. If the temperature rises too quickly compared to the increase in density, the cloud will become Jeans-stable, cease to collapse, and fail to form a star.

Fortunately, efficient cooling mechanisms are present in today's interstellar medium. Most notable among these is cooling by dust grains. Hot gas molecules can collide with dust grains and heat them conductively. The dust grains can then radiate away this energy by thermal emission of infrared radiation (provided the cloud is optically thin to this infrared radiation.)

The cloud can stay Jeans-unstable if the timescale for cooling is shorter than the timescale for gravitational contraction (the latter is the same as the heating timescale). That is, the collapse can proceed nearly isothermally rather than adiabatically.

(a) What is the timescale for gravitational contraction of a cloud of mass density  $\rho$ ? This is the time it takes for the cloud to decrease its radius by about a factor of 2.

A quick and crude way to estimate this time is to ask how long it takes a test particle to orbit just above the surface of a uniformly dense body of mass  $M$  and radius  $R$ . You can choose to use this orbit time as the contraction time (you will be off by a factor of a few), or you can do it your own way.

Express your answer in terms of  $G$  (the gravitational constant) and  $\rho$ .

From Kepler's Third Law, the orbital period is  $2\pi/\sqrt{GM/R^3}$ . Write  $M = 4\pi\rho R^3/3$  to write the orbital period as  $\sqrt{3\pi/G\rho}$ .

Actually, there is a slightly better estimate that is slightly more difficult to derive. The gravitational collapse time of the sphere is equal to half of the orbital period of an extremely elongated ellipse whose major axis equals  $R$ . Imagine a test particle on the surface of the sphere. As the cloud collapses, the test particle always feels the full gravitational force of the sphere that is beneath it. Thus, it is effectively falling in a point-mass (Kepler) potential. Its nearly radial trajectory is equivalent to a Kepler ellipse whose eccentricity is nearly unity, and whose apocenter equals  $R$  and whose pericenter equals 0. The semi-major axis of such an extremely elongated ellipse is  $R/2$ . It takes half a Kepler period to go from apocenter to pericenter. Thus, the collapse time

is  $t_{grav} = \pi/\sqrt{GM/(R/2)^3} = \sqrt{3\pi/32G\rho}$ .

(b) Express symbolically the mean thermal speed of  $H_2$  molecules in the cloud,  $v$ . Hint: Set the translational kinetic energy of a hydrogen molecule equal to  $kT$ , where  $k$  is Boltzmann's constant and  $T$  is the kinetic temperature of the molecules.

$$v = \sqrt{\frac{2kT}{\mu_g}}$$

(c) What is the timescale for cooling? This is the time for a hot gas molecule to collide with a dust grain. Assume canonical parameters for the interstellar medium today: dust is in the form of spheres of radius  $r = 0.1 \mu\text{m}$ , each sphere has an internal mass density  $\rho_p \approx 1 \text{ g cm}^{-3}$  (this is of order the density of ordinary, uncompressed bulk matter in the universe), and the volumetric mass density (in space) of dust is  $\rho_d \approx Z\rho$ , where  $Z \approx 10^{-2}$ .

Assume the velocity of dust grains is small compared to the velocity of hydrogen molecules.

Express your answer symbolically in terms of  $r$ ,  $\rho_d$ ,  $T$  (the temperature of the gas molecules),  $\mu_g$  (the mass of a gas molecule),  $\rho_p$  (the internal mass density of a dust grain), and  $k$  (Boltzmann's constant).

The mean free time between collisions between a gas molecule and a dust grain is  $t_{cool} = 1/n\sigma v$ , where  $n$  is the number density of dust grains,  $\sigma = \pi r^2$  is the cross-section for collision with a dust grain, and  $v = \sqrt{kT/\mu_g}$  is the mean speed of gas molecules. Note that we are neglecting the speed of the dust grain (it is assumed to be moving subsonically with respect to the gas), and we are neglecting the cross-sectional area of a gas molecule, which is tiny ( $[3 \text{ Angstroms}]^2$ ) compared to the dust grain. Write  $n = \rho_d/(4\pi\rho_p r^3/3)$  to write the cooling time as  $t_{cool} = 4\sqrt{\mu_g\rho_p}r/3\sqrt{kT\rho_d}$ .

(d) Initially, the cloud is initially marginally Jean-unstable; that is, its mass (radius) is just slightly above the Jeans mass (Jeans radius). At this initial time, which is shorter, the cooling time or the contraction (heating) time?

Note that I have not given you a temperature  $T$ . Use the fact that the cloud is marginally Jeans-unstable to solve for  $T$ , if you find you have to.

From the given parameters,  $\rho = 1.6 \times 10^{-20} \text{ g cm}^{-3}$ . This gives  $t_{grav} = 1.6 \times 10^{13} \text{ s}$  or about  $5 \times 10^5 \text{ yr}$ .

Now to evaluate  $t_{cool}$ , we need to know the temperature of the gas. The problem states that the cloud is initially marginally Jeans-unstable, so that means that  $t_{grav} \approx 2R/c_s$ , where the right-hand-side equals the sound-crossing time of the cloud ( $c_s = \sqrt{kT/\mu_g}$  is the sound speed in gas). Solve for  $c_s = 2R/t_{grav} = 4 \times 10^4 \text{ cm/s}$ , which means

that  $T = \mu_g c_s^2 / k = 36 \text{ K}$ , where we have used  $\mu_g = 2m_H = 3.4 \times 10^{-24} \text{ g}$ . Plugging in  $T$ ,  $\mu_g$ ,  $r = 1 \times 10^{-5} \text{ cm}$ ,  $\rho_p = 1 \text{ g cm}^{-3}$ , and  $\rho_d \approx 10^{-2} \rho = 1.6 \times 10^{-22} \text{ g cm}^{-3}$ , we get  $t_{cool} \approx 2 \times 10^{12} \text{ s} \approx 7 \times 10^4 \text{ yr}$ .

So initially, (b) is shorter than (a) by 1 order-of-magnitude.

(e) Explain why you might or might not expect to maintain the inequality of timescales found in (d) as the cloud collapses. Is this calculation promising for the formation of stars?

Do we expect (b) to remain shorter as the cloud continues to collapse? Dividing our expression for (b) by (a) gives a quantity which scales as  $(1/\sqrt{T\rho})/(1/\sqrt{\rho}) \propto 1/\sqrt{T\rho}$ . Now  $\rho$  is increasing as the cloud collapses. We have proven that initially,  $t_{cool} < t_{grav}$  so that  $T$  does not increase adiabatically as the cloud starts to collapse. Thus, we expect that as time advances,  $1/\sqrt{T\rho} \propto 1/\sqrt{\rho}$  continues to decrease. In other words, yes, the inequality is maintained. This is promising for star formation because it means that the cloud becomes increasingly Jeans-unstable and thus collapses under its own gravity unimpeded by thermal pressure.

## Problem 2. Lipstick

Estimate the mass in lipstick that Brittny Spears has ingested over her life so far. Express in whatever units seem most meaningful to you.

I don't wear lipstick but I do wear lip balm in winter months. I used to wear it more often than I do now, to the detriment of my current health. When I wore lip balm ("Chapstick") in college, I remember I would run out in about 1 month. From the dimensions of a Chapstick—a cylinder 0.6 cm in radius and 6 cm in height—I estimate a mass of 6 g, assuming a density of 1 g/cc. Over 12 months, the amount worn would be 72 g, which if I ate 50% of it—because of food and a bad habit of licking my lips when they got too dry—means I would eat 36 g in 1 year.

Now Brittny Spears is 23 years old (in fact, she was born in December 1981, a fact that many of you recalled with terrifying ease). I'll estimate that Brittny has been wearing lipstick for 15 years. Then if her lipstick is the same as my lip balm, she would have eaten about 500 g of lipstick so far. Probably Brittny puts on at least 2 coats a day, so that's about 1 kg of lipstick over her life so far.

This question was inspired by my recent stay in Singapore, during which I saw (1) many Brittny Spears posters, and (2) a poster at the airport which said, "The average woman eats 4.5 pounds (10 kg) of lipstick over a lifetime." My estimate of 1 kg per 15 years is in the ballpark, but it seems a tiny bit too low. When we account for a full human lifetime of about 80 years, then I would have estimated that Brittny would eat about 7 kg, not 10 kg, over her future life. Sources of error: density of lipstick is

probably closer to 2 g/cc, and she probably puts on more than 2 coats per day.

In any case, my experience wearing lip balm during the winter months gave me an answer that is apparently correct to order-of-magnitude, though I have not checked the source of the poster.