Astro 162 – Planetary Astrophysics – Solution to Problem Set 3

Problem 1. *Disk Heaven*

Consider once again the minimum-mass solar nebula, a circumstellar disk of gas and dust of solar composition orbiting the young sun.

X-ray satellite observations of young stars reveal that they are much more luminous in X-rays than the present-day sun. Some of these X-rays will strike the circumstellar disk and ionize its outermost layers of gas.

Assume this ionized outer layer of hydrogen is everywhere at a temperature of $T = 10^4\,\mathrm{K}$*. (One could compute its temperature from first principles by equating the heating rate due to photoionization to the cooling rate due to radiative de-excitation of collisionally excited ions, but that would be a calculation appropriate to a class on the interstellar medium.)*

(a) What is the thermal speed of ions in this hydrogen plasma? Express in [km/s].

The rule of thumb I remember is that at 100 K, hydrogen atoms have a sound speed of 1 km/s. Since the sound speed $c_s \propto \sqrt{T}$, at 10000 K, the sound speed should be about $\boxed{c_s \approx 10\,\mathrm{km/\,s}}$.

(b) At what critical disk radius, r_{crit}*, does the vertical scale height,* h*, of this* $T = 10^4\,\mathrm{K}$ *plasma equal the disk radius? In other words, at what critical disk radius does the aspect ratio,* h/r*, of the plasma disk equal 1? Answer symbolically in terms of* G*,* $M_\odot$*,* k*,* T*, and* μ *(the usual symbols;* μ *is the mass of the hydrogen ion), and numerically in [AU]. Remember that* h *is determined by hydrostatic balance in the vertical direction; we derived an expression for the vertical scale height in class, which you may either use or derive yourself.*

The scale height of material having sound speed c_s equals $h = c_s/\Omega$, where Ω is the local Keplerian angular frequency. Set $h = r$ to find that at r_{crit}, $v_K = c_s$ (the local Kepler velocity equals the sound speed). Since $v_K = \sqrt{GM_\odot/r}$, and since $c_s = \sqrt{kT/\mu}$, $\boxed{r_{crit} = GM_\odot\mu/kT}$. Plugging in numbers, I get $\boxed{r_{crit} = 10\,\mathrm{AU}}$.

(c) Imagine that you are a parcel of plasma at disk radius $r = r_{crit}$ *and hovering above the disk midplane at the full scale height* $z = h$*. Calculate the escape speed from the sun at your position, and compare to your thermal speed. Would you expect to remain hydrostatically bound to the sun?*

The escape speed at heliocentric distance d equals $v_{esc} = \sqrt{2GM_\odot/d}$. At $z = h = r$ and $r = r_{crit}$, $d = \sqrt{2}r_{crit}$. Then $\boxed{v_{esc} = \sqrt{\sqrt{2}kT/\mu}}$. This $\boxed{\text{barely exceeds the sound speed}}$, which equals $c_s = \sqrt{kT/\mu}$.

(d) Repeat (c) but at $r \gg r_{crit}$ to explain quantitatively why plasma might or might not remain bound to the sun at those great distances. Repeat also for $r \ll r_{crit}$.

At $r \gg r_{crit}$, the escape speed decreases, but the thermal speed of the plasma (which depends only on T) stays the same. We would expect plasma at $\boxed{r \gg r_{crit}}$ to be $\boxed{\text{blown away}}$. The technical term for this is $\boxed{\text{photoevaporation}}$.

At $\boxed{r \ll r_{crit}}$, the escape speed increases above the thermal speed, so that gas remains $\boxed{\text{hydrostatically bound}}$.

(e) Comment on the relevance of your answers above to the metallicity gradient with heliocentric distance in the giant gaseous planets. In other words, can you understand why Jupiter at 5 AU is closest to solar composition but Neptune at 30 AU is heavily enriched with metals?

$r_{crit} = 10\,\mathrm{AU}$ sits right on the orbit of Saturn. At distances inside Saturn's orbit, we expect photoionized hydrogen to remain hydrostatically bound to the sun. At distances outside Saturn's orbit, we expect photoionized hydrogen to photoevaporate and escape. Outside Saturn's orbit, the hydrogen escapes, leaving only the heavier elements (which have lower thermal speeds). In this way, the metallicity of outer disk material which eventually accretes to form Uranus and Neptune increases to super-solar values, while the metallicity of inner disk material which eventually accretes to form Saturn and Jupiter remains nearly solar (which it is).

Congratulations—you have recovered the essence of a classic paper by Shu, Johnstone, & Hollenbach that seeks to explain the heliocentric metallicity gradient among the giant planets.

Problem 2. *Terrestrial Rain Fall*

(a) Estimate the mean annual rain fall [in cm] that would result if all the solar radiation intercepted by the earth went into evaporating water.

The solar flux of radiation on the earth is $L_\odot/(4\pi r^2)$. Multiply by the cross-sectional area of the earth to get a power absorbed: $L_\odot \pi R_\oplus^2/(4\pi r^2)$. Divide by latent heat of vaporization of water to get the mass of water vaporized per year, $L_\odot \pi R_\oplus^2/(4\pi r^2 L_{vap})$, where $L_{vap} = 2 \times 10^{10}\,\mathrm{erg/g}$ is the latent heat of water. Assume that this vapor falls back to earth as rain, and divide by the surface area of the earth to get the mass column [g/cm^2] of rain per second, $L_\odot/(16\pi r^2 L_{vap})$. Divide by the mass density of water to get a thickness for the shell of rain that falls to the earth per second, and multiply by the number of seconds in one year to get the thickness of the shell of rain over one year: $L_\odot \pi \times 10^7/(16\pi r^2 L_{vap}\rho_{water}) = \boxed{555\,\mathrm{cm}}$ on rainfall in one year. This seems like a bit of an overestimate, but not by many orders of magnitude.

(b) List 2 factors that would reduce your estimate in (a).

1. About 30% of the incident solar radiation is reflected back into space by clouds, land, and solid particulates in the atmosphere.

2. About 20% of the incident radiation is absorbed by the atmosphere, most notably O_2 and O_3.

3. Only 70% of the earth is covered by oceans. So only 70% of the incident radiation that succeeds in making it all the way to the surface actually hits water.

If we account for these 3 factors, we should multiply our estimate by (100% - 30% - 20%) × 70% = 35%. That takes us down to 190 cm of rainfall in one year. In fact about 55 cm of rainfall fell over the entire earth in 1979. So our crude estimates aren't too bad, considering we have not modelled anything like ocean circulation patterns.

Problem 3. *Forming Terra*

Is gravitational focussing required to form the Earth?

Give a yes or no answer and justify it quantitatively. Credit awarded for care and precision in estimation.

$\boxed{\text{No,}}$ the last doubling time WITHOUT gravitational focussing takes about 10^8 years, which is short enough time to form the Earth within the age of the solar system. The doubling time WITHOUT gravitational focussing is given by $t_D = M/\dot{M} \sim R\rho_p/\Sigma\Omega$ as we derived in lecture. Because the last doubling time requires the longest time, to order-of-magnitude it is the time it takes to form Earth (all of the previous doublings went much quicker).

The only tricky part of this calculation is determining what Σ is appropriate. For the minimum mass solar nebula $\Sigma = 3000$ g/cm^2 at 1 AU. However, we are only worried about the rocky material, since the Earth has nearly zero (hydrogen/helium) gas content. (The gas might actually be gone—accreted onto the central star—by the time the Earth forms. If the gas is still present at the time of Earth assembly, it introduces aerodynamic drag forces and tidal torques that can complicate our simple analysis.) We can estimate Σ by taking the final mass of the Earth and smearing it over an annulus centered at 1 AU with a size $\Delta r \sim 1$ AU.

$$\Sigma = \frac{M_E}{2\pi r \Delta r} \sim \frac{6 \times 10^{27}\ \text{g}}{2\pi (1.5 \times 10^{13}\ \text{cm})^2} \sim 5\ \text{g/cm}^2$$

Using this value for Σ, $R = R_E/2$, $\rho_p \sim 1$ g/cm^3 and $\Omega \sim (1\text{yr})^{-1}$ you'll find that the $\boxed{\text{doubling time without focussing is } 6 \times 10^7 \text{ years}}$, which is a short enough time to form the Earth within the age of the solar system (4×10^9 years).

Problem 4. *Live Aluminum-26 in the Ancient Solar System*

In 1976, Lee et al. (1976, Geophys. Rev. Letters) published the results of isotope analyses *of Aluminum (Al) and Magnesium (Mg) in a calcium-aluminum inclusion (CAI) in the Allende chondritic meteorite. Different minerals in the CAI (fassaite, melilite, anorthite-B, anorthite-G) gave different isotope ratios. But together these ratios formed a trend: higher abundances of* ^{26}Mg *were found with higher abundances of* ^{27}Al*. The data looked something like this:*

MINERAL	$^{27}Al/^{24}Mg$	$^{26}Mg/^{24}Mg$
Fassaite	*0*	*0.1400*
Melilite	*10*	*0.1405*
Anorthite-B	*133*	*0.1467*
Anorthite-G	*240*	*0.1520*

Here $^{27}Al/^{24}Mg$ *is the number abundance of* ^{27}Al *atoms relative to the number abundance of* ^{24}Mg *atoms. The same notation applies to* $^{26}Mg/^{24}Mg$*. You can verify that these 4 points make a pretty good line.*

Now it is known that ^{26}Al *is an unstable isotope of Aluminum that beta-decays with a half-life of a mere* 0.7×10^6 yr *into* ^{26}Mg*. Thus, some of the* ^{26}Mg *found in the CAI may be radiogenic (it came from* ^{26}Al*). By contrast,* ^{24}Mg *is non-radiogenic and is used here as a reference. Furthermore,* ^{27}Al *is completely stable.*

Use arguments similar to those used in U-Pb dating (as discussed in class), and the data above, to estimate the $^{26}Al/^{27}Al$ *abundance ratio at the time the CAI solidified. (This problem is easy; please don't think too complicatedly.)*

We may see later that live ^{26}Al *was a major heat source in ancient planetesimals, able to melt their interiors.*

First recognize that all of the Al-26 is now gone; the CAI is 4.56 billion years old, and the half-life of Al-26 is a piddling 700,000 yr. Therefore the Mg-26 that we observe today equals ALL of the Al-26 that was present in the past, PLUS whatever Mg-26 was there to begin with (the primordial Mg-26).

How do we derive the primordial Mg-26? The data tell us how. Note that in the mineral fassaite, there is no measurable Al-27. Since the basic principle underlying all of radiogenic dating is that "isotopes move together but elements do not," we can assume that if no Al-27 was present in fassaite, that no Al-26 was there either. That means that in fassaite, whatever Mg-26 is present is primordial. The table tells us fassaite has Mg-26/Mg-24 = 0.1400. Now we can subtract this amount from the Mg-26/Mg-24 observed in another mineral, say melilite, to derive the pure radiogenic Mg-26/Mg-24 in melilite. This gives Mg-26/Mg-24 = 0.0005. Now melilite has Al-27/Mg-24 = 10. Then the radiogenic Mg-26/Al-27 = (Mg-26/Mg-24)/(Al-27/Mg-24) = 0.0005 / 10 = 5×10^{-5}. This, of course, equals the abundance of (now-dead) Al-26/Al-27 = $\boxed{5 \times 10^{-5}}$.

This may seem like a tiny amount, but it's enough to melt large bodies, as we will see in a future problem set!