Astro 162 – Planetary Astrophysics – Solutions to Set 4

Problem 1. *Rounding Rocks*

Estimate, to order-of-magnitude and from first principles, the critical size for rocks in space above which they would be nearly spherical.

Consider a rectangular block of rock of height L sitting on a sphere of the same material of radius R. The system as a whole would be considered "nearly spherical" if $L/R \ll 1$. We are free to choose how round is round; let's take $(L/R)_{crit} \equiv f_{crit} = 0.01$. The tallest rectangular block that can support itself is one for which the pressure at the base of the block is just below the yield stress; i.e., $\rho g L < Y$, where ρ is the density of rock, $g = GM/R^2$ is the gravitational acceleration at the surface of the sphere of mass M and radius R, and Y is the yield stress of rock. Probably we should take not Y but $Y/2$ so that there is a safety margin for stability. Write $M = 4\pi\rho R^3/3$ to write

$$\begin{aligned} \rho g L &= Y/2 \qquad (1) \\ \frac{4\pi}{3} G\rho^2 R L &= Y/2 \qquad (2) \end{aligned}$$

or $L = (3/8\pi)Y/G\rho^2 R$. But for roundness, we also want $L/R < f_{crit}$. Insert the latter condition into the former to derive $(3Y/8\pi G\rho^2)R^{-2} < f_{crit}$, or $\boxed{R > (3Y/8\pi G\rho^2 f_{crit})^{1/2}.}$

Let's take $Y = 10^8\,\mathrm{dyne\,cm^{-2}}$ (value appropriate to strong stony meteorites, as given in class), $\rho = 3\,\mathrm{g\,cm^{-2}}$, and $f_{crit} = 0.01$. Then $\boxed{R > 400\,\mathrm{km}}$. This accords well with observations; the asteroid Eos has dimensions of order 30 km and it is definitely not round, but rather looks like a banana; the largest asteroid Ceres has dimensions of order 1000 km and Hubble Space Telescope images reveal a spatially resolved disk.

Problem 2. *Collisional Cascades*

Consider an ensemble of bodies having a differential size distribution that takes the form of a power law:

$$\frac{dN}{dR} \propto R^{-q}\,. \qquad (3)$$

Here dN is the differential number of objects having radii between R and $R + dR$, and q is a constant. Note the minus sign in the above equation, and the fact that the equation is a proportionality (not an equality!)

It is known that for these bodies, the yield stress (having units of energy per volume, or equivalently force per area) is

$$Y \propto R^s \tag{4}$$

where s is another numerical constant. For this problem, interpret the yield stress Y as the stress required to catastrophically disrupt a target of radius of R.

(a) Repeat Dohnanyi's derivation of a catastrophic collisional cascade in quasi-steady-state to derive q as a function of s. Use the same assumption of Dohnanyi's, that the relative velocity between colliding bodies is constant.

As a check on your answer, verify that you obtain $q = 7/2$ for $s = 0$.

Following the derivation in lecture,

$$\dot{M} \sim n_{\text{target}} n_{\text{proj}} \sigma v_{\text{rel}} m_{\text{target}} \,. \tag{5}$$

Now we need to find how $\dot{M}$ scales with r_{target}. Everything is the same as in lecture except for n_{proj}, since now $Y \propto r^s$. The minimum projectile radius required to disrupt a target is found as

$$\frac{1}{2} m_{\text{proj}} v_{\text{rel}}^2 \sim Y \frac{4}{3} \pi r_{\text{target}}^3 \tag{6}$$

$$\rightarrow r_{\text{proj}} \propto r_{\text{target}}^{1+(s/3)} \,. \tag{7}$$

Therefore $n_{\text{proj}} \propto r_{\text{proj}}^{1-q} \propto r_{\text{target}}^{(1-q)[1+(s/3)]}$, whence

$$\dot{M} \propto r_{\text{target}}^{1-q} r_{\text{target}}^{(1-q)[1+(s/3)]} r_{\text{target}}^2 r_{\text{target}}^3 \,. \tag{8}$$

In steady-state, the entire set of exponents must sum to zero. Solve for q in terms of s: $\boxed{q = (21+s)/(6+s)}$.

(b) For bodies that are held together by self-gravity rather than by chemical bonds, what is s? Thereby infer q for such "rubble piles."

The yield stress is an energy density. For bodies held together by self-gravity, the relevant energy density is the gravitational binding energy of the object, $E \sim GM^2/R$, divided by the volume of the object, R^3. Then $Y \sim E/R^3 \propto R^2$. Hence $\boxed{s = 2}$. Plugging into part (a), we find that $\boxed{q = 23/8}$, which is more shallow than Dohnanyi's $q = 7/2$.

Problem 3. *Spin City*

Examine Figure 1 of the class hand-out from Hikida and Mizutani on "Tensile Strengths of Asteroids Predicted from the Relation of Asteroid Size and Spin Period."

Give a quantitative, order-of-magnitude derivation of the apparent rotation period limit of ~2 hours.

Explain the reasoning behind your derivation. You don't need to write an essay, but just writing down equations without any explanation will not suffice.

The minimum rotational period of the asteroid is a limit on how fast it can spin before breaking apart. The forces that come into play in this situation are centrifugal force, gravity and whatever tensile strength the asteroid has. Let us first think about the situation where the asteroid is held together mostly by its tensile strength. In this case, you might imagine, the minimum rotational period will depend strongly on the size of the asteroid, small objects will be able to spin much faster than big ones as long as the tensile strength does not depend on their size (which seems reasonable as the tensile strength is a property of the material they are made out of). This is not exactly what you see if you look at Hikida and Mizutani. Over 5 orders of magnitude in the size of the asteroid, the minimum period changes by about a factor of 2 or 3. Using this as an excuse to leave the tensile strength out of our derivation, let's see what we calculate for the minimum period using just gravity and centrifugal force.

$$\frac{GMm}{R^2} \sim \frac{mv^2}{R} \tag{9}$$

$$\frac{GM}{R^2} \sim \frac{4\pi^2 R}{P^2} \tag{10}$$

$$\frac{G(\frac{4\pi}{3}\rho R^3)}{R^2} \sim \frac{4\pi^2 R}{P^2} \tag{11}$$

$$P \sim \sqrt{\frac{3\pi}{G\rho}} \tag{12}$$

Plugging in a density of 5 g/cm^3 the minimum spin period is $\boxed{P \sim 1.5 \text{ hours}}$ which is very close to what is observed and is independent of the size of the asteroid, much like what we see in the figure. This result suggests that the tensile strength of asteroids is low, leading to the "rubble pile" explanation for the minimum spin period.

Problem 4. *How We Know What We Know About the Kuiper Belt*

Objects orbiting the Sun both reflect and absorb sunlight. The fraction of sunlight that a body reflects is given by its albedo, A. The rest is absorbed and heats the object. The absorbed energy is emitted as thermal radiation.

Kuiper belt objects (KBOs), orbiting at distances beyond the outermost planet Neptune, *are so distant from the Earth that resolved images of them are rarely possible. Most of them are, to us, point sources in both reflected and emitted light.*

(a) An astronomer wishes to measure the reflected flux from a KBO. At what wavelengths should the astronomer work, and why?

$\boxed{\text{Optical}}$, since the spectrum of the Sun peaks in the optical, and since we have plenty of optical telescopes on the planet.

(b) Astronomer X succeeds in measuring the reflected flux. Call this reflected flux, F_R. This flux is the amount of energy from the object striking a unit area of a detector at Earth per unit time. Assume the astronomer measures F_R when the object is at opposition—i.e., the Sun, Earth, and the KBO lie along a single line.

Write down a symbolic expression for F_R in terms of the distance to the object (d), the luminosity of the sun ($L_\odot$), the radius of the assumed spherical object (R), and the albedo (A).

Hint: reflected is italicized for a reason. If $d \gg 1$ AU (as it is for KBOs), how does F_R scale with d? The answer is not the usual inverse square law.

Light diverges twice—once on the way from the Sun to the KBO, and again on the way from the KBO to the Earth. On the first leg, the power reflected by the KBO equals

$$\frac{L_\odot}{4\pi d^2} A\pi R^2 \tag{13}$$

On the return leg, the reflected power is emitted into a hemisphere of area $2\pi(d-d_E)^2$, where $d_E = 1\,\text{AU}$ is the distance between the Earth and the Sun:

$$F_R = \frac{L_\odot}{4\pi d^2} A\pi R^2 \frac{1}{2\pi(d-d_E)^2} \tag{14}$$

$$\boxed{F_R = \frac{L_\odot A R^2}{8\pi d^2 (d-d_E)^2}} \tag{15}$$

For $d \gg d_E$, $F_R \propto 1/d^4$ which is much steeper than the usual inverse square law.

(c) Astronomer Y measures the emitted flux of the same object. The emitted flux is the re-radiated flux due to absorbed sunlight. Call this emitted flux, F_E.

Repeat (b), but for F_E.

Everything is the same as in (b) except that we replace the reflected fraction, A, with the absorbed fraction, $1-A$, and we say that the emitted flux is emitted isotropically into all 4π steradians by the uniform temperature KBO (rather than 2π steradians for the reflected flux):

$$\boxed{F_E = \frac{L_\odot(1-A)R^2}{16\pi d^2(d-d_E)^2}} \tag{16}$$

(d) At opposition, the KBO is observed to move relative to the distant (fixed) background stars with a proper motion of $\mu = 3$ arcseconds/hr. This proper motion is due purely to parallax. Estimate d based on this information, and express in AUs.

The instantaneous motion of μ is due mostly to the Earth's motion around the Sun; the (distant) KBO's velocity is much smaller than the Earth's velocity. The Earth is moving at $v_E \approx 30\,\mathrm{km\,s^{-1}}$. Call the distance between the Earth and the KBO, d_{EK}. Then

$$\mu = \frac{v_E}{d_{EK}} \tag{17}$$

Plugging in, we find $\boxed{d_{EK} \approx 49.5\,\mathrm{AU}}$.

Many of you under-estimated this distance because you took the full seasonal motion of the Earth (over several months). But the problem tells you only that the instantaneous (at opposition) proper motion is μ. You cannot assume that this instantaneous proper motion holds over the entire orbital period of the Earth. Indeed, μ is measured at the telescope over the course of a single night.

(e) Given your answers for (c) and (d), at what wavelengths should astronomer Y be observing? If you were unable to estimate d, assume $d = 40$ AU.

Assume the KBO radiates like a blackbody. Then set the blackbody luminosity of the KBO

$$\sigma T^4 4\pi R^2 \tag{18}$$

equal to the absorbed power

$$\frac{(1-A)L_\odot \pi R^2}{4\pi d^2} \tag{19}$$

to solve for the temperature

$$T \sim \left[\frac{(1-A)L_\odot}{16\pi\sigma d^2}\right]^{1/4} \tag{20}$$

The temperature is not terribly sensitive to the unknown A. If we assume $A = 0.1$, then $T \sim 38\,\mathrm{K}$. Using the Wien peak law, we find that the blackbody spectrum peaks at $\boxed{\lambda \sim 0.008\,\mathrm{cm} \sim 80\,\mu\mathrm{m}}$, which is in the infrared. In fact, the newly launched Spitzer Space Telescope is observing KBOs as I write this.

(f) Assume astronomers X and Y both know d. Can astronomer X, working alone, figure out R? Can astronomer Y, working alone, figure out R? Can astronomers X and Y, working together, figure out R?

Explain your reasoning for all parts.

$\boxed{\text{Working alone, neither}}$ astronomer can figure out the size of the KBO, since individually each astronomer has 1 equation in 2 unknowns, A and R.

$\boxed{\text{Working together, they can}}$ figure out A and R, since together they would have 2 independent equations in 2 unknowns.