

Astro 162 – Planetary Astrophysics – Solutions to Set 5

Problem 1. *Deep Impact*

Estimate, to order-of-magnitude, the critical size of the meteor plunging into the Earth's atmosphere that would fail to be slowed down significantly by air. That is, rocks of this size would strike the ground at nearly the full initial velocity of about $\sim 30 \text{ km s}^{-1}$ (the orbital speed of the earth around the sun). Assume the rock is made of solid, competent iron and does not shatter in the atmosphere.

The intrepid can try to do this on their own. The more cautious can work through the following steps:

(a) Imagine the asteroid plunging through the atmosphere. The atmospheric density decreases exponentially with height above the ground. The atmospheric density at sea level and the atmospheric scale height are given in Purcell's cheat sheet. Let's say the rock is initially located at a height z above the ground of 10 atmospheric scale heights, $z_0 = 10h$, and is headed straight down with an incoming velocity of $|v_0| = 30 \text{ km/s}$.

Write down an equation governing the deceleration, dv/dt , of the rock. Use what you think is the appropriate drag law, as given in class. Your equation should just read, $dv/dt = [\text{TERM}]$, where $[\text{TERM}]$ is a single term that depends only on the density of air (which depends on z), the radius of the rock, the internal density of the rock, the velocity v , and some dimensionless quantities of order unity. I would leave all variables in symbolic form and plug in numbers only at the very end!

Neglect the extra acceleration due to gravity. This is safe to do for the problem of interest, because in the time that the asteroid takes to fall from $z = 10h$ to $z = 0$, gravity only adds to the rock's already frightening velocity by a tiny fraction. You can check this easily, if you wish.

As discussed in class, 6 out of 8 times, the aerodynamic drag force on a particle of cross-sectional area A flying through gas of mass density ρ at relative speed v will just read $F_D = (1/2)C_D\rho v^2 A$, where $C_D \approx 0.5$ is a drag coefficient which never deviates by more than an order of magnitude from 1. We can rest assured that this is the correct drag law because the speed $v = 30 \text{ km/s}$ is supersonic; it exceeds the sound speed in air, which is about 0.3 km/s as given by Purcell's sheet. By contrast, the other 2 drag laws as discussed in class are appropriate for subsonic speeds.

Measure the height above the ground as $z > 0$. Then the downward velocity of the meteorite $v < 0$. Newton's law gives $m(dv/dt) = |F_D|$, where we have inserted an absolute value operator to ensure that the drag force always increases the velocity (makes the velocity less negative). Replace $m = 4\pi\rho_p R^3/3$, $A = \pi R^2$, and adopt an isothermal, exponential atmosphere, $\rho = \rho_0 \exp(-z/h)$ so that our equation reads

$$\boxed{\frac{dv}{dt} = -\frac{3}{8}C_D \frac{\rho_0 \exp(-z/h)}{\rho_p R} v^2} \quad (1)$$

Note that from Purcell's sheet, $\rho_0 = 10^{-3} \text{ g/cm}^3$ and $h = 8 \text{ km}$.

(b) Convert your equation to a linear, first-order differential equation. That is, you should have an equation which reads something like $dy/dx = f(x)y$, where $f(x)$ is some function of x . (Hint: begin by noting that $dt = dz/v$. If you have the correct drag law, note that you should substitute $y = v^2$.)

Do like the guy says and write $dt = dz/v$ to write $dv/dt = vdv/dz = (1/2)d(v^2)/dz$. Substitute $y = v^2$ to re-write equation (1) as

$$\boxed{\frac{dy}{dz} = \frac{3C_D \rho_0}{4\rho_p R} \exp(-z/h)y} \quad (2)$$

(c) Solve the equation, and impose the boundary conditions to solve for any constants of integration. You can use the math hand-out in class if you wish. You should have an explicit equation for $v(z)$ (velocity as a function of height).

Replace $3C_D \rho_0 / 4\rho_p R \equiv C$, a constant. The equation then reads

$$\frac{dy}{dz} - C e^{-z/h} y = 0 \quad (3)$$

The integrating factor for this linear ODE is $\lambda = \exp(-\int C e^{-z/h} dz) = \exp(C h e^{-z/h})$. (Yes, you might have never thought you'd see one, but we are talking about an exponential of an exponential.) Multiply (3) by λ ,

$$e^{C h e^{-z/h}} \left[\frac{dy}{dz} - C e^{-z/h} y \right] = 0 \quad (4)$$

The left-hand-side, thanks to our integrating factor, is now easily integrable. Integrate with respect to z to find

$$y e^{C h e^{-z/h}} = D = y_0 e^{C h e^{-z_0/h}} \quad (5)$$

where we have set the constant of integration, D , equal to a form that permits easy insertion of the initial condition. Solve for y :

$$\boxed{y = y_0 e^{C h (e^{-z_0/h} - e^{-z/h})}} \quad (6)$$

(d) Calculate the critical radius of the iron meteorite for which $v(z = 0) = 0.9v(z_0)$, i.e., the size of the meteor that would only have its initial velocity reduced by 10%. Do bigger things hit the ground faster or do smaller things hit faster according to your solution? Express your answer in [m].

First recognize that $e^{-z_0/h} = e^{-10} \ll 1$. Then at $z = 0$, $y(z = 0) = y_0 e^{-Ch}$. Since $y = v^2$, and the problem asks us to consider $v(z = 0) = 0.9v(z_0)$, $y(z = 0)/y_0 = 0.9^2 = 0.8 = e^{-Ch}$. Solve for $C = -\ln(0.8)/h$. Since C is a function of R , solve for $R = -3C_D \rho_0 h / 4 \rho_p \ln(0.8)$. Take $\rho_p = 7$ [cgs] (iron), $C_D = 0.5$, $\rho_0 = 10^{-3}$ [cgs], and $h = 8$ km, to find $R = 200$ cm = 2 m. Bigger things hit the ground faster than smaller things.

(e) For bonus points, give a physical interpretation of this critical radius as expressed symbolically. Hint: think column density.

We could have avoided all this math and reasoned roughly as follows: the slow-down time of the rock is its momentum divided by the drag force. This is of order $t \sim mv/F_D \sim \rho_p R / \rho v$. Over this time, the rock travels $l \sim vt \sim \rho_p R / \rho$. The mass column density of air that it encounters is $\rho l \sim \rho_p R$. Recognize $\rho_p R$ to be the mass column density of the rock itself! Then the rock slows down significantly if it encounters a column density of air equal to its own column density. Most of the column density of the earth's atmosphere is in the last scale height before you hit the ground; the column density of the atmosphere is therefore of order $\rho_0 h$. Set this equal to $\rho_p R$ and solve for $R \sim \rho_0 h / \rho_p \sim 100$ cm, which is pretty close to our more precise answer in (d). Masses larger than this never encounter a column density of air that exceeds their own internal column density.

Problem 2. Your Mileage May Vary

Using what you know about aerodynamic drag and the energy content of gasoline, estimate the mileage rating of a typical suburban sedan driving at highway speeds. Express in miles per gallon of gas. Explain your reasoning throughout. Give a symbolic expression in terms of whatever variables you need, and a numerical estimate.

We assume that at highway speeds, the dominant source of energy dissipation is air drag. That is, we need to burn gasoline fast enough to balance the amount of power dissipated through air drag. First we establish the relevant air drag law. Since the size of the car greatly exceeds the mean free path in air, and since the speed of the car is less than the speed of sound, we are either in the turbulent ram pressure regime or the viscous regime. The deciding factor is the Reynolds number of the flow around the vehicle. The Reynolds number is

$$Re \sim \frac{Rv}{\nu} \sim \frac{2 \text{ m} \times 60 \text{ mph}}{0.2 \text{ cm s}^{-1}} \sim 3 \times 10^6 \quad (7)$$

which definitely greatly exceeds the critical number required for turbulence, ~ 100 . Therefore we are in the turbulent ram pressure regime. We got the viscosity of air, $\nu \sim 0.2 \text{ cm}^2 \text{ s}^{-1}$, from Purcell's cheat sheet.

The work required to travel distance x is $F \times x$, where $F = (1/2)C_D \rho v^2 A$ is the drag force. Since any real car engine is inefficient, the actual work required will be $1/f$ times Fx , where f is the efficiency of the engine. We set this equal to the energy content of the gasoline that we burn, $G \times u$, where G is the mass of gasoline burned, and u is the energy per unit mass of gasoline. Then

$$\frac{1}{f}Fx \sim Gu \tag{8}$$

$$x/G \sim \frac{fu}{F} \sim \frac{fu}{0.5C_D \rho v^2 A} \tag{9}$$

where x/G is the distance travelled per mass of gasoline burned. Let's first estimate u . In class we established that the energy content of food (cheese) was about 10^{11} erg/g . Let's say gasoline is about the same as cheese (fat), maybe 2 times more energy-packed because of the industry practice of "cracking" (attaching more hydrogen bonds to the carbon chain to yield more energy). So $u \sim 2 \times 10^{11} \text{ erg/g}$.

Now A is the face-on cross-sectional area of the car. I estimate this is about $A \sim 2 \text{ m}^2$. At highway speeds, $v \sim 60 \text{ mph} \sim 2.7 \times 10^3 \text{ cm s}^{-1}$. We take $C_D \sim 1$. The density of air is $\rho \sim 10^{-3} \text{ g/cm}^3$. The engine efficiency might be $f \sim 25\%$. Putting it all together,

$$x/G \sim 700 \text{ cm/g} \tag{10}$$

Now since there are 4 liters per gallon, and 1000 grams per liter of water, and since gasoline is like water, there are 4000 grams of gasoline per gallon of gasoline. There are 1.6 km per mile, and 10^5 cm per km . Putting in all these conversion factors into our expression for x/G , we get 17 miles per gallon, which is about right. My Volkswagen Passat (read: BMW for the people) gets about 30 miles per gallon.

Problem 3. Hot Tiles

Using what you know about aerodynamic drag (and other physics), estimate the peak temperature of the tiles on the bottom of the Space Shuttle during re-entry. Give a symbolic expression in terms of whatever variables you need, and a numerical estimate.

Our experience with problem 2 tells us that not too far above the top of the first atmospheric scale height, $z \sim h$ (that is, for ρ not too much less than 10^{-3} g/cm^3), the shuttle is in the turbulent ram pressure regime. The bottom of the shuttle experiences

an air drag force per unit area of $(1/2)C_D\rho v^2$. Multiplying this force by the velocity gives us the rate of power dissipation. A fraction, say $f \sim 0.5$, of this power goes into heating the tiles. Let's say the tiles radiate like blackbodies; then

$$\sigma T^4 \sim f \frac{1}{2} C_D \rho v^3 \quad (11)$$

$$T \sim \left(\frac{f C_D \rho v^3}{2\sigma} \right)^{1/4} \quad (12)$$

At $z \sim h$, $\rho \sim 10^{-3} \text{ g/cm}^3$. The only question is, what is v ? Now an absolute maximum of v is given by the orbital velocity of the shuttle, which is $v_{\text{max}} \sim \sqrt{GM_{\oplus}/R_{\oplus}} \sim 8 \text{ km s}^{-1}$, where we have used the fact that the altitude of the shuttle above the surface of the Earth is small compared to the radius of the Earth. If we insert $v \sim v_{\text{max}}$ into our expression for T , we would get $T_{\text{max}} \sim 4 \times 10^4 \text{ K}$.

In fact, the shuttle does not experience this peak temperature for very long. This is because the shuttle almost immediately slows down to its terminal velocity right when it hits $z \sim h$. Now our experience with problem 1 suggests that since the characteristic size of the Shuttle $\gg 2 \text{ m}$, that the shuttle might never be slowed. But the Shuttle is built like an airplane—its area-to-mass ratio is much greater than that of an ordinary, solid rock, because it has giant wings. I looked up the dimensions and the mass of the Shuttle. The bottom of the Shuttle is a triangle of base 79 feet and height 122 ft. Then its area is about $A \sim 5 \times 10^6 \text{ cm}^2$. The weight of the Shuttle (orbiter component only) is $m \sim 2 \times 10^9 \text{ g}$. The Shuttle slows (reduces its velocity by about a factor 2) when it runs into its own mass in air. It needs to run into a length of air, l , given by

$$\rho A l \sim m \quad (13)$$

$$l \sim 4 \text{ km} \quad (14)$$

before its velocity slows appreciably. Travelling at speed v_{max} , it takes $\sim l/v_{\text{max}} \sim 0.5 \text{ s}$ to slow down. Since this is much shorter than the time it takes to land, we conclude that the Shuttle readily slows down to its terminal velocity.

The terminal velocity, as derived in class, is

$$v \sim \sqrt{\frac{mg}{0.5C_D\rho A}} \sim 0.3 \text{ km s}^{-1} \quad (15)$$

which is smaller than v_{max} . Plugging v into (12), we find $T \sim 3000 \text{ K}$. Note that v is small enough that the landing takes $h/v \sim 24 \text{ s}$ if the Shuttle fell straight down. In

practice, the Shuttle comes in at an angle, so we estimate the landing takes $3h/v \sim 72$ s, or about a minute. This accords with my experience of watching Shuttle landings on TV.

Problem 4. *When the Dust Settles*

Imagine a dust particle having a radius $r = 1 \mu\text{m}$ located initially one gas scale height, $z = h$, above the midplane of the minimum-mass solar nebula at a heliocentric radius of $a = 1 \text{ AU}$. The solar nebula has the full complement of hydrogen gas of surface density $\Sigma \sim 1.5 \times 10^3 \text{ g/cm}^2$. In the absence of turbulence, the dust particle will fall vertically through the gas due to the vertical component of the star's gravity (we dealt with this vertical component in class). Assume the dust particle does not move radially as it falls (this is a fine approximation for small dust particles).

Estimate, to order-of-magnitude, how long it takes for this dust grain to fall vertically to the midplane ($z = 0$). Give a symbolic answer in terms of whatever variables you need, and a numerical estimate in [yr]. In particular, specify how does the settling time scales with r and with a , for $a > 1 \text{ AU}$.

The settling time you have computed is relevant for interpreting the spectra of disks. Old disks in which the dust has settled are less luminous than young disks in which the dust has not settled, because settled (flat) disks absorb less central starlight than unsettled (flared) disks.

The grain achieves terminal velocity. The terminal velocity is derived by balancing the gravitational force against the drag force. The vertical gravitational force at height z above the midplane of the disk, as derived in class, is $m\Omega^2 z$, where Ω is the Keplerian angular frequency of orbital motion. Here $m \sim 4\rho_p r^3$ is the mass of the grain, $\rho_p \sim 2 \text{ g/cm}^3$ is the grain's internal density, and r is the radius of the grain.

The drag force is in the Epstein, or free molecular drag regime. This is because the grain size, $r \sim 1 \mu\text{m}$, is much smaller the mean free path in gas, $\lambda \sim (n\sigma)^{-1} \sim 3 \text{ cm}$. Here $n \sim \rho/\mu_g \sim 3 \times 10^{14} \text{ cm}^{-3}$ is the number density of hydrogen molecules in the minimum-mass solar nebula, $\rho \sim 10^{-9} \text{ g cm}^{-3}$ is the mass density of hydrogen molecules, and $\mu_g \sim 3 \times 10^{-24} \text{ g}$ is the mass per hydrogen molecule. The cross-section for a hydrogen molecule to collide with another molecule is $\sigma \sim (3 \text{ Angstroms})^2 \sim 10^{-15} \text{ cm}^2$. Furthermore, we can bet that the terminal velocity will be much less than the speed of sound, and check our bet later.

We set the force of gravity equal to the Epstein drag force to solve for the terminal velocity, v :

$$4\rho_p r^3 \Omega^2 z \sim \rho v c_s (\pi r^2) \tag{16}$$

$$v \sim \frac{\Omega^2 z \rho_p r}{\rho c_s} \quad (17)$$

where c_s is the thermal speed of gas molecules. Notice that the terminal velocity v depends on height, z . The higher above the midplane we are, the faster the grain falls. The height-halving time is

$$t_{1/2} \sim \frac{z/2}{v} \sim \frac{\rho c_s}{2\rho_p r \Omega^2} \quad (18)$$

which is independent of z . Plugging in numbers— $c_s \sim 1 \text{ km s}^{-1}$, $\Omega \sim 2\pi/(1 \text{ yr})$ —we find that $t_{1/2} \sim 10^5 \text{ yr}$. Strictly speaking, to fall all the way from some height z to $z = 0$ takes an infinite number of height-halving times (Zeno's paradox). In practice, we might be interested in going, say, from $z \sim h$ to $z \sim 10^{-2}h$, which takes about 7 height-halving times ($2^7 = 128$). So a better answer might be $t \sim 7 \times 10^5 \text{ yr}$. It is readily verified by plugging in numbers into (17) that $v < c_s$, which validates our use of the Epstein drag formula.

The time to fall scales as $t_{1/2} \propto 1/r$; bigger grains fall faster by their mass-to-area ratio. The time to fall is independent of heliocentric distance, a . This is seen as $t_{1/2} \propto \rho c_s / \Omega^2 \propto a^{-2.8} a^{-0.2} / a^{-3} \propto a^0$.