Astro 162 – Planetary Astrophysics – Solutions to Set 6

Problem 1. *Photon Pinball*

This problem shows how the usual $e^{-\tau}$ attenuation factor for light through an absorbing medium is incorrect for light passing through a cloud.

Idealize the cloud as a uniform, 1-dimensional slab comprising particles that can only SCATTER light. That is, a photon passing through a cloud gets bounced like a pinball from cloud droplet to cloud droplet, preserving its frequency and never getting absorbed by any droplet. A few photon/pinballs are lucky enough to make it through the cloud, but most get pinballed back out the way they came. We will calculate the fraction that make it through.

Take the cloud to have a droplet density [droplets per cubic volume] η, the droplet radius to be R, and the vertical thickness of the cloud to be z_{max}. Measure vertical distance through the cloud by z, where the top of the cloud is located at $z = 0$ and the base of the cloud is located at $z = z_{max}$.

(a) Write down the optical depth of the cloud, τ.

The optical depth for a uniform density slab is equal to the number density of scatterers times the cross-section per scatterer times the thickness of the slab.

$$\boxed{\tau = \eta \pi R^2 z_{max}} \tag{1}$$

(b) Incident photons from the sun strike the top of the cloud. The photons have a number flux, F_i [number per time per area]. What is the number density of incident photons at the top of the cloud? Call this photon number density n_i. These incident photons have NOT been scattered yet by any droplet. Remember that flux is a number density multiplied by a speed.

$$\boxed{n_i = F_i/c} \tag{2}$$

(c) These photons pinball/random walk through the scattering droplets. Random walks are described by the diffusion equation,

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial z^2} \tag{3}$$

where D is the diffusion coefficient and n is the photon number density.

Express D in terms of symbols defined above and whatever fundamental constants you deem appropriate.

The diffusion coefficient, or the diffusivity, is, to order of magnitude, just the mean free path of a single photon times the speed of a photon. The mean free path of a single photon travelling through the scattering medium is $1/(\eta\pi R^2)$. The speed is c. So $\boxed{D = c/(\eta\pi R^2)}$.

(d) In steady-state, $\partial n/\partial t = 0$ (the number density of photons everywhere in the cloud does not change with time). Write down the solution to the diffusion equation for $n(z)$. You should have two, as yet unknown, constants of integration.

The general solution to $\partial^2 n/\partial z^2 = 0$ is $\boxed{n = Az + B}$, where A and B are constants.

(e) To solve for the two constants of integration, you need two boundary conditions. The first condition is that $n_t = n(z = z_{max})$. Here n_t is the number density of photons at the base of the cloud. These photons comprise the transmitted flux.

The second condition is that the (net, number) flux, F, of photons at $z = 0$ equals the incident flux, F_i (directed down into the cloud) MINUS the outgoing, reflected flux, F_r (directed up, away from the cloud into space). Recall Fick's law, which is just another way of writing the diffusion equation, that the (net) flux $F = -D\partial n/\partial z$. So we have $F(z = 0) = F_i - F_r$.

Use the above, and the fact that the incident flux, F_i, must equal the reflected flux, F_r, PLUS the transmitted flux, F_t, to calculate T, the ratio of the transmitted flux to the incident flux, in terms of τ.

At $z = 0$, there are TWO contributions to the local photon number density. The first contribution is from incident photons, n_i. The second contribution is from photons reflected out of the cloud, n_r. Therefore, at $z = 0$,

$$n = n_i + n_r = B \tag{4}$$

Now $F_i = F_r + F_t$ means that $n_i = n_r + n_t$ (just divide by c). Use the latter equation to write $n_r = n_i - n_t$ and substitute into (4) to solve for

$$B = 2n_i - n_t \tag{5}$$

At $z = z_{max}$, $n = n_t = Az_{max} + B$. Use this equation and equation (5) to solve for

$$A = \frac{2(n_t - n_i)}{z_{max}} < 0 \tag{6}$$

Now, the net flux $F = -D\partial n/\partial z = -DA$. Notice the net flux is constant with height, as it must be lest the photon density increase or decrease somewhere in the cloud, violating our steady-state assumption. At $z = z_{max}$, $F = F_t = -DA$. Then $n_t = -DA/c$. Note that since $A < 0$, $n_t > 0$ which is correct (you cannot have a negative number density). Thus,

$$
\begin{aligned}
T &\equiv n_t/n_i && (7)\\
&= -\frac{DA}{cn_i} && (8)\\
&= -\frac{2D(n_t - n_i)}{z_{max} c n_i} && (9)\\
&= -\frac{2(T-1)}{\tau} && (10)
\end{aligned}
$$

Finally, solve for T in terms of τ:

$$\boxed{T = \tfrac{2}{2+\tau}} \tag{11}$$

which is certaintly not exponentially sensitive to optical depth. And thank goodness, or else it would be plenty darker on the Earth on cloudy days. Note our expression has the right limits: the transmission goes to 1 if $\tau = 0$, and goes to zero (linearly) as τ goes to infinity.

(f) Evaluate T for the clouds of sulfuric acid on Venus. You may use the review article on Venus handed out in class to find numbers for η, R, and z_{max}.

From the notes, $\eta \sim 10^2\,\mathrm{cm}^{-3}$, $R \sim 5\,\mu\mathrm{m}$, and $z_{max} \sim 10\,\mathrm{km}$, from which we calculate $\tau \sim 80$. Then $\boxed{T \sim 0.025}$. Just a few percent of the light from the sun makes it past the clouds and on to the Venusian surface, yet this few percent is enough to make Venus is a hellish 700 K.

Problem 2. *DIFFUSION: A New Fragrance by 162*

Someone opens a bottle of unusually strong perfume in a classroom.

(a) If the perfume molecules travel only by random walking (diffusion), how long does it take for the entire room to smell of perfume?

If the room has length L and the mean free path of a perfume molecule is λ then it will take $\left(\frac{L}{\lambda}\right)^2$ steps for the perfume molecules to cross the room. Each step takes $t = \lambda/v_{th}$ where v_{th} is the thermal velocity of the particles, given by $v_{th} = (2kT/\mu)^{1/2}$.

Thus, the time it takes for the perfume molecules to make it to the other side of the room is:

$$t = \frac{L^2}{\lambda^2}\frac{\lambda}{v_{th}} = \frac{L^2}{\lambda v_{th}} \tag{12}$$

If we take a look at Purcell's cheat sheet, you will see that the mean free path in air is 7×10^{-6} cm and the thermal velocity is 4×10^4 cm/s. If we have a room that is 7 meters long, the time for the perfume molecules to cross the room is $\boxed{2 \times 10^6 \text{ seconds}}$, or about a month.

(b) Does your answer in (a) accord with your experience? If not, why not?

This answer definitely does not accord with my experience. If diffusion were the only thing happening, it would take us 23 days to smell the perfume. I think it takes probably a few minutes, maybe even a few seconds, for the perfume to get across the room. What is not taken into account by our estimate is macroscopic circulation of air in the room. We're walking around, opening doors, breathing, and generally stirring the bathtub. The perfume can travel with these wind currents much more quickly then it can random walk its way across the room.

Problem 3. *Melting Vesta*

As described in class, Vesta is an asteroid whose interior may have differentiated chemically, judging from the light basaltic crust that we know to be present on the surface of Vesta. To have differentiated, Vesta must have been hot enough to melt a substantial fraction of its interior.

Assume Vesta was assembled very early in the solar system's history, so early that it contained the full complement of live radioactive Aluminum-26 inferred to be present from primitive meteorites. For those of you who worked out problem 3 of last week's set, you know that the full complement means a $^{26}Al/^{27}Al$ *number abundance ratio of* 5×10^{-5} *(*^{27}Al *is the common, non-radioactive isotope of Al.) The radioactive decay of each* ^{26}Al *nucleus into* ^{26}Mg *releases a gamma-ray photon having 1.8 MeV (mega-electron-volts). The exponential decay timescale is short,* 7×10^5 yr, *compared to the age of the solar system.*

(a) Assume for this part that all of the energy released by the radioactive decay of ^{26}Al *is trapped in Vesta. Estimate the fraction of Vesta that melts due to this short-lived radionuclide. Is* ^{26}Al *promising as a heat source?*

(You will need to make a reasonable model for the relevant relative elemental abundances in Vesta. You may use the table of solar abundances by Grevesse and Sauval, but remember that while the relative metal abundances of Vesta may be close to those of primitive meteorites and therefore close to those of the Sun, Vesta is very much unlike the Sun in lacking H and He—it's a rock, not a ball of gas!)

Let's take Vesta to be made of silicates like olivine = $MgFeSiO_4$, which has a mean molecular weight of $24 + 56 + 28 + 64 = 172$ times the mass of hydrogen. There are $N_{olivine} \sim (4/3)\pi R^3 \rho/(172 m_H) \sim 1.6 \times 10^{45}$ olivine molecules in Vesta, where we have used $\rho = 5\,\mathrm{g/cm^3}$ and $R = 288\,\mathrm{km}$.

Now how many (normal, atomic weight 27) aluminum atoms are there? We can use the table of Grevesse and Sauval, but the table is useful only if we choose a particular abundance ratio of aluminum to some other element. Which element do we choose: Mg, Fe, Si, or O? Let's choose the element that will limit the number of olivine molecules we can make. From Grevesse and Sauval, there are more than 10 oxygen atoms for every Mg, Fe, or Si atom, and since $MgFeSiO_4$ uses up only 4 oxygen atoms per (Mg,Fe,Si), the number of olivine molecules we can make is limited by abundances of (Mg,Fe,Si) and not by oxygen. The least abundant element among (Mg,Fe,Si) is Fe (but not by much). So Fe is the limiting element, whose abundance relative to even less abundant elements (like Al) we will take to be solar.

The number of Fe atoms is equal to $N_{olivine}$. From Grevesse and Sauval, we find there are 10 Fe atoms for every 1 ^{27}Al atom. And we know there were 5×10^{-5} ^{26}Al atoms for every ^{27}Al atom. So the number of ^{26}Al atoms in the past was $5 \times 10^{-5} \times 0.1 \times 1.6 \times 10^{45} \sim 8 \times 10^{39}$.

The decay of all this ^{26}Al yields $8 \times 10^{39} \times 1.8 \times 10^6 \times 1.6 \times 10^{-12}\,\mathrm{erg} = 2.3 \times 10^{34}\,\mathrm{erg}$. It takes about 2×10^{10} erg to heat 1 gram of rock from 300 K to the melting temperature of about 1500 K and then to melt it into liquid. So we can heat and melt about 10^{24} g of Vesta. The mass of Vesta is about 5×10^{23} g. So $\boxed{100\%}$ of Vesta could have melted!

The main uncertainty in our calculation was the percentage of (normal) ^{27}Al of Vesta by mass. Some geologists in the class estimated this to be 0.1%, as opposed to our $0.1 m_{Al}/m_{olivine} \sim 1.5\%$. If we use 0.1%, we find that about $\boxed{13\%}$ of Vesta can melt, which is still substantial.

(b) Justify quantitatively the assumption in (a) that most of the energy released by ^{26}Al is trapped in Vesta and that little of it gets radiated away into space. Hint: think about turkeys.

We estimate the thermal conduction time for a turkey of radius 10 cm to be 3 hours. Vesta is 288 km in radius. Heat obeys the diffusion equation, so it takes 3 hours $\times (288\,\mathrm{km}/10\,\mathrm{cm})^2 \sim 9 \times 10^{16}\,\mathrm{s} \sim 3 \times 10^9\,\mathrm{yr}$ for heat to conduct through Vesta. $\boxed{\text{Three Gyr is plenty longer than the decay time of } ^{26}\text{Al}}$, so you can think of all of the radiogenic heat as being generated and trapped in Vesta with zero loss.