Astro 162 – Planetary Astrophysics – Solutions to Set 7

Problem 1. *Air Conditioners, Airplanes, and Atmospheric Stability*

Airplanes cruise at altitudes of 10 km above sea level. The outside air at this altitude is at a pressure $p \approx 264$ millibars, and at a temperature $T \approx 223$ K.

(a) Would the air outside the airplane, if adiabatically compressed, be suitable for use in a plane's passenger cabin? Take the ratio of specific heats of air to be $\gamma = c_p/c_v = 7/5$, appropriate for classically excited, rigid diatomic rotating molecules.

$P \propto \rho^\gamma \propto \rho^{7/5}$. By the ideal gas law, $P \propto \rho T$. Combining the two proportionalities gives $T \propto \rho^{2/5}$. Then $P \propto T^{7/2}$.

Compress the air outside the airplane from $P = 0.264$ bar to $P = 1$ bar. Then the temperature increases from 223 K to $223 \times (1/0.264)^{2/7} = 326\,\mathrm{K}$. This is $\boxed{\text{much too hot}}$ for humans onboard the airplane.

(b) Based only on your answer in (a), would you conclude that the Earth's atmosphere below 10 km altitude is stable or unstable to convection? In other words, is the actual temperature gradient in the lower atmosphere sub-adiabatic or super-adiabatic?

In part (a) we effectively computed the adiabatic temperature gradient—from the airplane's altitude (at 0.264 bar) to sea level (at 1 bar), the adiabatic temperature gradient would tell us that the temperature increases from 223 K to 326 K. Since the actual temperature at sea level (where humans work and play happily) is cooler than 326 K, we conclude that the actual temperature gradient is not as steep as the adiabatic temperature gradient. In other words, the actual gradient is $\boxed{\text{sub-adiabatic}}$ and therefore the atmosphere is $\boxed{\text{stable to convection.}}$ ("Stable to convection" means the atmosphere is NOT convecting.)

Problem 2. *Just How Tiny is the Tiny Superadiabatic Temperature Gradient?*

It is often stated that convection is so efficient at transporting heat that the actual temperature gradient in a convective atmosphere is only slightly greater than the adiabatic temperature gradient. Here we estimate quantitatively what "slightly greater" means.

Recall that the Brunt-Vaisala (B-V) frequency is the frequency of buoyant vertical motions in an atmosphere, and is given by

$$\omega_{B-V}^2 = \left[\frac{\partial T}{\partial z}|_{actual} - \frac{\partial T}{\partial z}|_{adiabatic}\right]\frac{g}{T}\,, \tag{1}$$

where $g > 0$ is the gravitational acceleration, T is temperature, and z is height above the *base of the atmosphere. Define*

$$\Delta\nabla T = \left[\frac{\partial T}{\partial z}|_{actual} - \frac{\partial T}{\partial z}|_{adiabatic}\right] \quad (2)$$

to be the difference between the actual temperature gradient and the adiabatic temperature gradient. We will estimate $\Delta\nabla T$*, and compare it to* $\nabla T|_{actual}$*. Remember that if* $\Delta\nabla T < 0$*, then* $\omega^2_{B-V} < 0$*—in other words, the B-V frequency has an imaginary component, any vertical motions are unstable, and convection ensues. Since* $\nabla T < 0$*,* $\Delta\nabla T < 0$ *means the absolute value of the actual temperature gradient,* $|\nabla T|_{actual}$*, exceeds the absolute value of the adiabatic temperature gradient,* $|\nabla T|_{adiabatic}$*; we say the actual temperature gradient is superadiabatic (but not by much) in convective atmospheres.*

(a) Consider a parcel of gas moving adiabatically upwards through a convective (superadiabatic) atmosphere. The parcel has mass density ρ *and specific heat [erg/(gram K)] at constant pressure* c_p*. It maintains pressure equilibrium with its surroundings: as it rises, the parcel's pressure matches exactly the surrounding atmospheric pressure (which is decreasing with increasing height). The parcel's temperature decreases adiabatically, while the atmosphere's temperature drops superadiabatically. In other words, the adiabatic drop in the parcel's temperature is not as much as the drop in the surrounding environment's temperature, because the actual temperature gradient of the environment is superadiabatic.*

After the parcel has risen length l*, where* l *is small compared to the pressure scale height, how much excess energy density [erg/cm*^{3}*] does the parcel carry relative to its surroundings? Use the variables given above. Call this extra energy density* ϵ*.*

After rising length l, the parcel is slightly hotter than its surroundings; call the temperature difference δT. Since the parcel is behaving adiabatically while the atmosphere is behaving "in actuality," $\delta T = |\Delta\nabla T| \times l$. This excess temperature gives rises to an excess energy density of $\boxed{\epsilon = |\Delta\nabla T| l\rho c_p}$.

(b) Give an approximate symbolic expression for the upward velocity, v*, of the buoyant parcel after it has travelled distance* l*. Remember that the parcel is unstably buoyant; it experiences an upward acceleration,* $(\delta\rho/\rho)g$*, where* g *is the local (downward) planetary gravitational acceleration. You should first understand why* $\delta\rho$*, the density difference between the parcel and its surroundings, is negative. Reduce your expression to one that DOES NOT contain* ρ *or* $\delta\rho$*, but DOES contain* T*.*

After rising length l, the parcel now has a kinetic energy per unit mass equal to the buoyant acceleration times the distance travelled. Then $v^2/2 = (\delta\rho/\rho)gl$.

Now we replace $\delta\rho/\rho$ in favor of $\delta T/T$. For an ideal gas, $\delta P/P = \delta\rho/\rho + \delta T/T$. But the parcel is assumed to be in pressure equilibrium with its surroundings, so that

$\delta P = 0$. Then $\delta\rho/\rho = -\delta T/T = -|\Delta\nabla T| l/T$.

Putting it all together, $\boxed{v = \sqrt{2|\Delta\nabla T| l^2 g/T}}$.

(c) Assume that convection dominates heat transport through the atmosphere. The atmosphere must transport the full flux of energy that is deposited at its base. Call this flux, F [erg cm^{-2} s^{-1}]. Therefore $F = \epsilon v$ (the density of any quantity times the speed with which that quantity moves gives you a flux).

Use $F = \epsilon v$ and (a) and (b) to solve for an approximate symbolic expression for $\Delta\nabla T$. Your answer should depend on F, l, T, g, c_p, and ρ.

Straightforward algebra at this point gives $\boxed{|\Delta\nabla T| = (F/\rho c_p l^2)^{2/3}(T/2g)^{1/3}}$.

(d) Calculate $|(\Delta\nabla T)/(\nabla T)_{actual}|$ for conditions appropriate to Jupiter's atmosphere at a pressure of 1 bar. For this numerical evaluation, the only quantity which is not given by data is l, the distance a parcel travels before it dissolves away into its surroundings. No one knows what l—the infamous "mixing length" of convection—is. It is reasonable that $l < h$, the pressure scale height of the atmosphere. Here let's charge forth boldly and use $l \sim h$.

For F, remember that the energy that is transported upwards through the atmosphere is absorbed energy from the sun, times a factor of $\sim$2 to account for the internal heat that Jupiter self-generates because it is still (after all these eons) gravitationally contracting and converting gravitational potential energy into heat.

Is your computed dimensionless quantity tiny? You may find Figure 1.26 of the hand-out helpful for some actual numbers.

Let's first compute $|\Delta\nabla T|$ using our expression in (c). First, $F \sim (L_\odot/4\pi(5\,\mathrm{AU})^2) \times 0.5 \times 2 \sim 6 \times 10^4$ [cgs], where we have accounted crudely for a Bond albedo via the factor of 0.5 and for internal energy generation via the factor of 2. From the aforementioned figure, at $P = 1$ bar, $T = 170$ K. By the ideal gas law, with mean molecular weight $\mu = 4 \times 10^{-24}$ g (appropriate for molecular hydrogen plus some heavier elements), $\rho = \mu P/kT = 2 \times 10^{-4}\,\mathrm{g/cm^3}$. Furthermore $c_p \sim 2k/\mu \sim 7 \times 10^7\,\mathrm{erg/g\,K}$, and $g = GM/R^2 = 3 \times 10^3$ [cgs]. Finally, $l \sim kT/\mu g \sim 2 \times 10^6$ cm. Putting it all together, $|\Delta\nabla T| \sim 3 \times 10^{-9}\,\mathrm{K/cm}$.

Now for the actual temperature gradient; from the aforementioned figure, the temperature drops about 40 K over 20 km in height. Then $|\nabla T|_{actual} \sim 2 \times 10^{-5}\,\mathrm{K/cm}$.

Thus, $\boxed{|\Delta\nabla T|/|\nabla T|_{actual} \sim 10^{-4}}$. Yes, this is pretty tiny.

Problem 3. *The Four Seasons*

Consider a simplified model for the seasonal variation of the temperature of an atmo-

sphere. Assume that the temperature of each vertical column is independent of height and neglect horizontal heat transport. In this scheme, the instantaneous temperature as a function of time, $T(t)$, is determined by the absorption of a variable solar flux, $F(t) = F_0 + F_1 \cos \omega t$, where F_0 and F_1 are constants. The re-radiated flux into space is $F_{IR}(t) = \sigma T^4$. Here $\omega = 2\pi/P$ is the frequency over which the solar flux varies at some latitude on the obliquely tilted planet, and P is the orbital period of the planet about the sun.

(a) Describe why

$$\int_0^\infty dz \; \rho c_p \frac{\partial T}{\partial t} = F(t) - \sigma T^4 \tag{3}$$

where c_p is the specific heat [erg/g/K] of atmospheric gas, z is the height above the planet's surface, and ρ is the mass density of the atmosphere. You should explain the physical significance of each term.

The left-hand-side is the time rate of change of the thermal heat content of the atmosphere per unit area. On the right-hand-side, F is the source term: the amount of energy per unit area per time that is incident upon the atmosphere. The loss term is σT^4, which is the flux that is radiated by the atmosphere into space.

(b) Linearize this equation to obtain

$$I \frac{\partial T_1}{\partial t} + 4\sigma T_0^3 T_1 = F_1 \cos \omega t \tag{4}$$

where $I = \int_0^\infty dz \rho c_p$ is the thermal inertia, $T = T_0 + T_1$ and $\sigma T_0^4 = F_0$. Here T_1, the (time-varying) perturbation temperature, is small compared to (the time-independent) T_0. "Linearize" means "Taylor expand T, drop the zeroth order terms, and keep only the first order perturbation terms."

The derivation is straightforward:

$$I \frac{\partial T}{\partial t} = F - \sigma T^4 \tag{5}$$

$$I \frac{\partial (T_0 + T_1)}{\partial t} = F - \sigma (T_0 + T_1)^4 \tag{6}$$

Note that T_0 is constant, Taylor expand $(T_0 + T_1)^4$, and write $F = F_0 + F_1 \cos \omega t$:

$$I \frac{\partial T_1}{\partial t} = F_0 + F_1 \cos \omega t - \sigma T_0^4 - 4\sigma T_0^4 \frac{T_1}{T_0} \tag{7}$$

Note that $\sigma T_0^4 = F_0$, so that

$$I\frac{\partial T_1}{\partial t} = F_1 \cos\omega t - 4\sigma T_0^3 T_1 \tag{8}$$

as desired.

(c) Try a solution of the form $T_1/T_0 = \Delta\cos(\omega t - \phi)$. Prove that

$$\tan\phi = \frac{\omega I T_0}{4F_0}\,, \tag{9}$$

and that

$$\Delta = \frac{\cos\phi}{4}\frac{F_1}{F_0}\,. \tag{10}$$

Plug and chug:

$$\begin{aligned}
-IT_0\omega\Delta\sin(\omega t-\phi) &+ 4\sigma T_0^4\Delta\cos(\omega t-\phi) = F_1\cos\omega t && (11)\\
-IT_0\omega\Delta\sin(\omega t-\phi) &+ 4F_0\Delta\cos(\omega t-\phi) = F_1\cos\omega t && (12)\\
-IT_0\omega\Delta(\sin\omega t\cos\phi - \cos\omega t\sin\phi) &+ 4F_0\Delta(\cos\omega t\cos\phi + \sin\omega t\sin\phi) && (13)\\
= F_1\cos\omega t & && (14)
\end{aligned}$$

Now the coefficients of the $\sin\omega t$ terms must match:

$$\begin{aligned}
-IT_0\omega\Delta\cos\phi + 4F_0\Delta\sin\phi = 0 && (15)\\
\rightarrow \boxed{\tan\phi = \tfrac{\omega I T_0}{4F_0}} && (16)
\end{aligned}$$

The coefficients of the $\cos\omega t$ terms must similarly match:

$$\begin{aligned}
IT_0\omega\Delta\sin\phi + 4F_0\Delta\cos\phi = F_1 && (17)\\
IT_0\omega\Delta\tan\phi + 4F_0\Delta = F_1 && (18)
\end{aligned}$$

Insert (16) into (18) to find:

$$\frac{I^2T_0^2\omega^2\Delta}{4F_0} + 4F_0\Delta = F_1/\cos\phi \tag{19}$$

$$4F_0\Delta[\frac{I^2T_0^2\omega^2}{(4F_0)^2} + 1] = F_1/\cos\phi \tag{20}$$

$$4F_0\Delta[\tan^2\phi + 1] = F_1\cos\phi \tag{21}$$

$$4F_0\Delta\sec^2\phi = \mathrm{F}_1\cos\phi \tag{22}$$

$$\rightarrow \boxed{\Delta = \frac{F_1\cos\phi}{4F_0}} \tag{23}$$

(d) State the physical significance of

$$\frac{IT_0}{4F_0}\,. \tag{24}$$

(Hint: recall lecture). Estimate the value of this quantity (be sure to give units) for a region on the Earth's surface at the latitude of, say, the Mojave desert.

The numerator is the thermal heat content of the atmosphere per unit area. The denominator (modulo the 4) is the flux that the atmosphere radiates. Thus the entire quantity is of order the time that it takes the atmospheric column of heat to radiate away its heat. This is the $\boxed{\text{thermal time}}$ of the atmosphere.

Estimate I for the atmosphere: $\rho = 10^{-3}\,\mathrm{g\,cm^{-3}}$, $c_p \sim 2k/\mu$, $\mu = 30m_H$, $m_H = 1.67\times10^{-24}\,\mathrm{g}$, atmospheric scale height $h = 8\,\mathrm{km}$. So $I \sim \rho c_p h \sim 4\times10^9$ (in cgs units). Also $F_0 = \sigma T_0^4$, and $T_0 \approx 280\,\mathrm{K}$. Therefore

$$\boxed{\frac{IT_0}{4F_0} \sim 0.03\,\mathrm{yr}} \tag{25}$$

(e) Estimate F_1/F_0 for the Mojave desert.

The Mojave desert lies at a latitude of 35 degrees north. At spring equinox, the solar flux striking the desert is proportional to $\cos(35°)$. At summer solstice, the flux is proportional to $\cos(35° - 23°)$. So $\boxed{F_1/F_0 \approx (\cos(12°) - \cos(35°))/\cos(35°) \approx 0.2}$.

(f) Evaluate ϕ and Δ using (d) and (e), and insert into your solution for the fractional seasonal temperature variation, T_1/T_0. Comment on the relation of your results to reality.

The frequency of flux variations is $\omega = 2\pi/1\,\mathrm{yr}$. Then $\tan\phi = 0.18$, or $\boxed{\phi = 10°}$. This means the temperature perturbation of the atmosphere, T_1, LAGS the driving force (the variation of solar flux with orbital phase) by 1 yr × 10°/360° ∼ 0.3 month. This is not too bad an approximation of reality; maximum temperatures in any given location on the Earth are reached about 1–2 months after summer solstice. Now $\boxed{\Delta = T_1/T_0 = (\cos(10°)/4) \times (F_1/F_0) = 0.05}$. So the change in temperature between springtime and summer on the desert is about 0.05 times 280 K or about 14 degrees Kelvin (Celsius). This also seems about right.

Problem 4. *Eternal Sunshine of the Venusian Mind*

Explain quantitatively why the Venusian surface does or does not experience the seasons over a Venusian year.

If you are going to use temperature as a variable, be clear about the distinction between actual temperature and effective temperature (in other words, be sure you understand the greenhouse effect).

Feel free to look up any numbers that you need.

We need to find the thermal time of Venus's atmosphere and compare it to a Venusian year. If the thermal time is longer than the year, the atmosphere does not have time to adjust to temperature changes during a year and the surface of Venus does not experience seasons. If the thermal time is short compared to a year, the atmosphere can adjust quickly enough that the surface will experience seasons.

As we discussed in class the thermal time of the atmosphere is given by the total heat content of a column of atmosphere divided by the rate at which it radiates away its energy.

$$t \sim \frac{3nkTh}{\sigma T_e^4} \tag{26}$$

In this expression h is the scale height of the atmosphere, T_e is the effective temperature and T is the temperature at the base of the atmosphere. The distinction between the effective temperature and the temperature at the base of the atmosphere is very important in the case of Venus, because the two are quite different due to the strong greenhouse effect. We showed in class that $T \sim \tau^{1/4} T_e$ so we can put this in for T in our equation for the thermal time of the atmosphere. Alternatively, you can recognize that nkT is just the pressure at the surface and rewrite the equation like so:

$$t \sim \frac{3Ph}{\sigma T_e^4} \tag{27}$$

Looking up values for Venus, ($P = 90$ bar at the surface, $h = 15$ km and $T_e = 230$ K) you can calculate that the thermal time of Venus's atmosphere is about $\boxed{\text{80 years}}$ which is much longer than a Venusian year.

This means that Venus's surface $\boxed{\text{does not experience seasons.}}$