Astro 162 – Planetary Astrophysics – Problem Set 8

Due Thursday March 31.

Readings: Skim all of Chapter 8. Pay particular attention to section 8.3 on "The Dynamic Earth and Drifting Continents," and the last part of section 8.5 on "Internal Heating from Core Separation."

Problem 1. Dry vs. Wet Adiabatic Lapse Rates

Here we derive and numerically evaluate the adiabatic temperature gradient for dry and moist atmospheres. Remember that these adiabatic gradients DO NOT EQUAL the actual gradients. The adiabatic gradients are reference quantities for any planetary atmosphere, to be compared against the actual gradient to decide whether an atmosphere is convective or not. The adiabatic gradients are derived under the assumption that each displaced fluid parcel exchanges zero heat with its surroundings, contrary to what happens in actuality.

(a) Use the adiabatic relation for (a dry, non-condensing) gas

$$P \propto \rho^{\gamma}\,, \tag{1}$$

the condition of hydrostatic equilibrium,

$$\frac{1}{\rho}\frac{dP}{dz} = -g \tag{2}$$

and the ideal gas law,

$$P = \frac{\rho k T}{\mu} \tag{3}$$

to solve for the so-called "dry adiabatic lapse rate," dT/dz, in terms of g, k, μ, and γ.

(b) Evaluate the dry lapse rate for conditions in the Earth's troposhere. Express in [K / km], and make sure you get the sign right.

(c) If, as the temperature decreases with increasing height above the atmosphere, some trace species (e.g., water) of the atmosphere begins to condense out and make clouds, latent heat is released by the condensing vapor. This latent heat increases the temperature of the atmosphere. Thus, the so-called "wet adiabatic lapse rate" is smaller than the "dry" rate; the temperature of the moist atmosphere still decreases with height, but less steeply than for a dry atmosphere because of the latent heat that is released by condensing vapor.

The first law of thermodynamics can accommodate this extra release of latent heat:

$$dU = -P\,dV + dQ + L_{vap}\,dm \tag{4}$$

where L_{vap} [erg/gram] is the latent heat of vaporization (condensation) of some vapor, and dm is the differential amount of vapor mass that condenses. Here U, P, and V are the internal energy, pressure, and volume of our proverbial test parcel of gas. By definition, for adiabatic processes, $dQ = 0$ (the parcel exchanges no heat with its environment).

Use the first law as written above, the condition of hydrostatic equilibrium, the ideal gas law, and the lecture regarding specific heats for gas to derive the wet adiabatic lapse rate,

$$\frac{dT}{dz} = \frac{-g}{\frac{k}{\mu}\frac{\gamma}{\gamma-1} - L_{vap}\frac{dw}{dT}} \tag{5}$$

where w is the mass of condensed vapor per unit mass of (total) atmospheric gas. DO NOT use the dry adiabatic relation given in (a), $P \propto \rho^{\gamma}$, because it does not apply under these moist conditions. DO use $\gamma \equiv C_P/C_V$; this is merely a definition for the symbol γ as used in the equation above for the wet adiabatic lapse rate.

Remember that we are treating the condensable vapor as a trace (minor) constituent of the total atmosphere. This is a fine approximation for many situations: for Jupiter, the condensable vapors are ammonia, water, and ammonium hydrosulfide (vs. molecular hydrogen and helium for the bulk of the atmosphere); for the Earth, the condensable is water (vs. nitrogen, oxygen, and carbon dioxide); for Venus, the condensable is hydrosulfuric acid (vs. nitrogen and carbon dioxide). The quantity dw/dT measures the amount of vapor mass that *condenses* per unit mass of atmospheric gas, per degree Kelvin change. If $dT < 0$, then $dw > 0$; hence $dw/dT < 0$.

(d) ESTIMATE the wet adiabatic lapse rate for conditions in the Earth's troposphere. Express in [K / km].

Assume that the air is completely saturated with water vapor at every height. Use the Clausius-Clapeyron formula for the saturation vapor pressure of water,

$$P_{sat,water} = C_L \exp[-L_s/(R_{gas}T)] \tag{6}$$

where $C_L = 3 \times 10^7$ bar, $R_{gas} = 8.3 \times 10^7$ erg K^{-1} mole^{-1}, and $L_s = 5.1 \times 10^{11}$ erg mole^{-1}. As T decreases, $P_{sat,vapor}$ decreases; in other words, as it gets colder, the air can't hold as much water vapor; the water vapor is forced to condense out as droplets (fog). Use this information and $P_{sat,water}$ to ESTIMATE dw/dT.

Check that you have the right magnitude for the wet lapse rate relative to the dry lapse rate. One should be larger than the other.

(e) Do you expect moist regions in the atmosphere where vapor is condensing to be more likely to be convective than dry regions in the atmosphere? Which environment do hang-gliders prefer? In the more vigorously convective environment, what is the extra source of energy?

Problem 2. The Lid Makes All the Difference

This problem asks you to estimate the thicknesses of the hard, conductive lithospheres for Venus, Earth, and Mars. In so doing, you will have taken a small step towards understanding why Earth exhibits plate tectonics while Venus and Mars do not.

Take the main internal source of energy of every planet to be from radioactive U, Th, and K (not ^{26}Al; that fuel source ran out a long time ago). Assume each planet is, on average, made of chondritic material which today emits $5 \times 10^{-8}\,\mathrm{erg\,s^{-1}\,g^{-1}}$. So given the total mass of each planet, you know the internal luminosity of each planet.

This radiogenic energy is carried conductively through the hard outer lithosphere of each planet. Recall the equation for the conductive heat flux:

$$F = -K_C \nabla T \tag{7}$$

where K_C is the thermal conductivity (proportional to, but having wildly different units from, the thermal diffusivity), and T is the temperature. Flux F has units of $\mathrm{erg\,s^{-1}\,cm^{-2}}$.

At the top of the lithosphere, the temperature is just the surface temperature. You know what that is. (And don't forget that Venus is plenty hotter than you would guess from the blackbody formula.)

At the bottom of the lithosphere (top of the convective mantle), the temperature is $\sim$1200 K, the temperature at which rock starts to become plastic (that is, flowing; the viscosity of rock is exponentially sensitive to temperature).

Use all the facts above to estimate the thicknesses of the lithospheres for Venus, Earth, and Mars.

Geologists tells us that "thin" lithospheres are not good for plate tectonics because they are too light and therefore do not subduct (sink) to greater depths (like trying to drown a rubber duck; it just won't go down.) Geologists also tell us that "thick" lithospheres are not good for plate tectonics because upwelling magma cannot penetrate to the surface. Can you begin to understand why the Earth exhibits tectonic motion but the other planets do not?

Problem 3. Sea Floor Spreading

The mid-Atlantic ridge represents an upwelling boundary between two convection cells. New material is flowing away from the ridge at a roughly constant velocity, u.

(a) Reproduce our order-of-magnitude symbolic expression for u and verify that it is about a few cm/yr.

(b) Assume Middle-Earth is a planet whose mantle dynamics is very similar to that of the Earth. The One Ring was dropped by Frodo into Mount Doom, where it was swept away into the interior of the planet.

If (contrary to legend) the One Ring can withstand any environmental condition, about how long would it take for the One Ring to re-appear?

(c) The conductive heat flux from the ocean floor decreases with distance away from the ridge. That is, your feet feel hotter if you are standing near the ridge than if you are standing far from the ridge. How does the heat flux, F, scale with distance, x, from the ridge?

A proportionality (scaling relation) is sufficient; there is no need to provide constants.

In fact, such a dependence with distance is measured, and is one of the key pieces of evidence we have that plate tectonics works the way we think it does.

Problem 4. Helium Rain, or More for Your Money

Saturn's total luminosity is about 1.7 times greater than the power that it absorbs from sunlight. It is thought that the gravitational settling of helium out of the gaseous envelope of Saturn onto the core of Saturn is responsible for this extra power. The pressure in the interior of Saturn is high enough that helium is a liquid. It is a liquid that is immiscible (doesn't mix) with hydrogen, which is also liquid. The liquid helium literally rains to greater depths within the planet, converting its gravitational potential energy to heat.

(a) If all of the helium from Saturn's gaseous envelope rains onto the surface of the rocky core of Saturn, how much energy would be released? Answer in [ergs]. An order-of-magnitude answer is sufficient; you don't need to use any complicated model for Saturn's interior structure. Just remember that Saturn is thought to have a rocky core having a mass of $\sim$10 Earth masses, and take Saturn's envelope to be of solar composition

(and yes, I really do mean solar composition this time!)

(b) How much excess energy is released by Saturn over the age of the solar system, assuming its luminosity has been constant for this time?

(c) Compare (a) to (b) and comment.