

Astro 162 – Planetary Astrophysics – Solutions to Set 8

Problem 1. Dry vs. Wet Adiabatic Lapse Rates

Here we derive and numerically evaluate the adiabatic temperature gradient for dry and moist atmospheres. Remember that these adiabatic gradients DO NOT EQUAL the actual gradients. The adiabatic gradients are reference quantities for any planetary atmosphere, to be compared against the actual gradient to decide whether an atmosphere is convective or not. The adiabatic gradients are derived under the assumption that each displaced fluid parcel exchanges zero heat with its surroundings, contrary to what happens in actuality.

(a) Use the adiabatic relation for (a dry, non-condensing) gas

$$P \propto \rho^\gamma, \quad (1)$$

the condition of hydrostatic equilibrium,

$$\frac{1}{\rho} \frac{dP}{dz} = -g \quad (2)$$

and the ideal gas law,

$$P = \frac{\rho k T}{\mu} \quad (3)$$

to solve for the so-called “dry adiabatic lapse rate,” dT/dz , in terms of g , k , μ , and γ .

Say $P = K\rho^\gamma$. Then this relation inserted into the condition of hydrostatic equilibrium gives $\gamma\rho^{\gamma-1}(K/\rho)d\rho/dz = -g$. Rewrite this using again $P = K\rho^\gamma$ to find $-g = (\gamma P/\rho)d \ln \rho/dz$.

Now use the equation of state: $P = K\rho^\gamma = \rho k T/\mu$. Divide both sides by ρ and take the log of both sides: $\ln K + (\gamma - 1) \ln \rho = \ln(k/\mu) + \ln T$. Combine this with the equation at the end of the previous paragraph to find the dry adiabatic lapse rate:

$$\boxed{\frac{dT}{dz} = -\frac{\gamma-1}{\gamma} \frac{g\mu}{k}} \quad (4)$$

Notice this relation is nicely independent of ρ , T , or P ; it’s a constant.

(b) Evaluate the dry lapse rate for conditions in the Earth’s troposphere. Express in $[K/km]$, and make sure you get the sign right.

For the Earth's troposphere, $\gamma = 7/5$, $g = 980$ [cgs], $\mu \approx 30 \times 1.67 \times 10^{-24}$ g. Then

$$\boxed{dT/dz \approx -10 \text{ K/km}}.$$

(c) If, as the temperature decreases with increasing height above the atmosphere, some trace species (e.g., water) of the atmosphere begins to condense out and make clouds, latent heat is released by the condensing vapor. This latent heat increases the temperature of the atmosphere. Thus, the so-called “wet adiabatic lapse rate” is smaller than the “dry” rate; the temperature of the moist atmosphere still decreases with height, but less steeply than for a dry atmosphere because of the latent heat that is released by condensing vapor.

The first law of thermodynamics can accommodate this extra release of latent heat:

$$dU = -P dV + dQ + L_{\text{vap}} dm \quad (5)$$

where L_{vap} [erg/gram] is the latent heat of vaporization (condensation) of some vapor, and dm is the differential amount of vapor mass that condenses. Here U , P , and V are the internal energy, pressure, and volume of our proverbial test parcel of gas. By definition, for adiabatic processes, $dQ = 0$ (the parcel exchanges no heat with its environment).

Use the first law as written above, the condition of hydrostatic equilibrium, the ideal gas law, and Tuesday's (April 1) lecture regarding specific heats for gas to derive the wet adiabatic lapse rate,

$$\frac{dT}{dz} = \frac{-g}{\frac{k}{\mu} \frac{\gamma}{\gamma-1} - L_{\text{vap}} \frac{dw}{dT}} \quad (6)$$

where w is the mass of condensed vapor per unit mass of (total) atmospheric gas. DO NOT use the dry adiabatic relation given in (a), $P \propto \rho^\gamma$, because it does not apply under these moist conditions. DO use $\gamma \equiv C_P/C_V$; this is merely a definition for the symbol γ as used in the equation above for the wet adiabatic lapse rate.

Remember that we are treating the condensable vapor as a trace (minor) constituent of the total atmosphere. This is a fine approximation for many situations: for Jupiter, the condensable vapors are ammonia, water, and ammonium hydrosulfide (vs. molecular hydrogen and helium for the bulk of the atmosphere); for the Earth, the condensable is water (vs. nitrogen, oxygen, and carbon dioxide); for Venus, the condensable is hydrosulfuric acid (vs. nitrogen and carbon dioxide). The quantity dw/dT measures the amount of vapor mass that condenses per unit mass of atmospheric gas, per degree Kelvin change. If $dT < 0$, then $dw > 0$; hence $dw/dT < 0$.

Insert the ideal gas law into the equation for hydrostatic equilibrium to derive:

$$\frac{k}{\mu\rho} \frac{d(\rho T)}{dz} = -g \quad (7)$$

$$\frac{dT}{dz} = -\frac{\mu g}{k} - \frac{T}{\rho} \frac{d\rho}{dz} \quad (8)$$

$$\frac{dT}{dz} = -\frac{\mu g}{k} - \frac{T}{\rho} \frac{d\rho}{dT} \frac{dT}{dz} \quad (9)$$

where for the last line we have used $d\rho/dz = (d\rho/dT) \times (dT/dz)$. Solve for dT/dz :

$$\frac{dT}{dz} = \frac{-\mu g/k}{1 + \frac{T}{\rho} \frac{d\rho}{dT}} \quad (10)$$

Now we use the first law of thermodynamics to derive $d\rho/dT$. From the lecture, we know that the differential change in the total energy of a system (here, a parcel of gas of fixed number of molecules) is given by

$$dU \equiv C_V dT \quad (11)$$

where C_V is the specific heat at constant volume. The above relation is the definition of C_V . We also know from lecture that

$$C_P = C_V + Nk \quad (12)$$

$$\gamma \equiv C_P/C_V \quad (13)$$

From the above 2 relations,

$$C_V = C_P - Nk \quad (14)$$

$$C_V = \gamma C_V - Nk \quad (15)$$

$$\rightarrow C_V = Nk/(\gamma - 1) \quad (16)$$

and therefore

$$dU = \frac{Nk}{\gamma - 1} dT \quad (17)$$

Insert this into the first law:

$$\frac{Nk}{\gamma - 1} dT = -P dV + L_{vap} dm \quad (18)$$

Divide by the total mass, M , of the parcel:

$$\frac{k}{\mu(\gamma - 1)} dT = -P d(\rho^{-1}) + L_{vap} dw \quad (19)$$

where $\mu \equiv M/N$, $V/M \equiv \rho^{-1}$, and $w \equiv m/M$. Expand $d(\rho^{-1})$, divide by dT , and re-arrange:

$$\frac{k}{\mu(\gamma - 1)} dT = +\frac{P}{\rho^2} d\rho + L_{vap} dw \quad (20)$$

$$\frac{d\rho}{dT} = \frac{1}{\gamma - 1} \frac{k \rho^2}{\mu P} - \frac{\rho^2}{P} L_{vap} \frac{dw}{dT} \quad (21)$$

The last equation combines with the ideal gas law to yield:

$$\frac{T}{\rho} \frac{d\rho}{dT} = \frac{1}{\gamma - 1} - L_{vap} \frac{dw}{dT} \frac{\mu}{k} \quad (22)$$

Insert this into (10) to find the desired relation:

$$\frac{dT}{dz} = -\frac{\mu g/k}{\frac{\gamma}{\gamma-1} - L_{vap} \frac{dw}{dT} \frac{\mu}{k}} \quad (23)$$

(d) *ESTIMATE* the wet adiabatic lapse rate for conditions in the Earth's troposphere. Express in $[K / km]$.

Assume that the air is completely saturated with water vapor at every height. Use the Clausius-Clapeyron formula for the saturation vapor pressure of water,

$$P_{sat,water} = C_L \exp[-L_s/(R_{gas}T)] \quad (24)$$

where $C_L = 3 \times 10^7$ bar, $R_{gas} = 8.3 \times 10^7$ erg K^{-1} mole $^{-1}$, and $L_s = 5.1 \times 10^{11}$ erg mole $^{-1}$. As T decreases, $P_{sat,vapor}$ decreases; in other words, as it gets colder, the air

can't hold as much water vapor; the water vapor is forced to condense out as droplets (fog). Use this information and $P_{sat,water}$ to ESTIMATE dw/dT .

Check that you have the right magnitude for the wet lapse rate relative to the dry lapse rate. One should be larger than the other.

Now dw/dT equals

$$\frac{dw}{dT} = -\frac{d}{dT} \left(\frac{P_{sat,water}}{P_{air}} \frac{\mu_{water}}{\mu_{air}} \right) \quad (25)$$

Note the inclusion of a minus sign; if $dT < 0$, the left-hand side must be negative ($dw > 0$; water is condensing), while $dP_{sat,vapor} < 0$. Assume that P_{air} is constant over the height interval over which water is condensing; this is likely to be a good assumption because the saturation vapor pressure is exponentially sensitive to temperature, and all of the water is likely to be condensed over a small change in temperature (i.e., a small change in altitude). Then

$$\frac{dw}{dT} = -\frac{\mu_{water}}{\mu_{air}} \frac{1}{P_{air}} \frac{d}{dT} P_{sat,water} \quad (26)$$

From Clausius-Clapeyron, $P_{sat,water} = C_L \exp[-L_s/(R_{gas}T)]$. Then

$$L_{vap} \frac{dw}{dT} \frac{\mu_{air}}{k} = -\frac{L_{vap} C_L L_s \mu_{water}}{P_{air} k R_{gas} T^2} \exp[-L_s/(R_{gas}T)] \quad (27)$$

Use $T = 280$ K, $P_{air} = 1$ bar, $R_{gas} = 8.3 \times 10^7$ erg K⁻¹ mole⁻¹, $C_L = 3 \times 10^7$ bar, $L_{vap} = 2 \times 10^{10}$ erg/g, $L_s = 5.1 \times 10^{11}$ erg mole⁻¹, $\mu_{water} = 18 \times 1.7 \times 10^{-24}$ g to find that

$$L_{vap} \frac{dw}{dT} \frac{\mu_{air}}{k} \approx -3 \quad (28)$$

Plug this value into (23) and evaluate as in (b) to find the moist adiabatic lapse rate,

$$\boxed{\frac{dT}{dz} \approx -6 \text{ K/km}} \quad (29)$$

or about half the dry rate.

(e) Do you expect moist regions in the atmosphere where vapor is condensing to be more likely to be convective than dry regions in the atmosphere? Which environment do hang-

gliders prefer? In the more vigorously convective environment, what is the extra source of energy?

Because the moist rate is less steep than the dry rate, and because convection occurs when the actual gradient exceeds the adiabatic rate, convection is more likely to occur under moist conditions because the moist adiabatic rate is easier to exceed. Physically, there is more energy available to drive convective motions in the atmosphere because of the latent heat released by condensation. Hang-gliders benefit from this extra energy.

Problem 2. *The Lid Makes All the Difference*

This problem asks you to estimate the thicknesses of the hard, conductive lithospheres for Venus, Earth, and Mars. In so doing, you will have taken a small step towards understanding why Earth exhibits plate tectonics while Venus and Mars do not.

Take the main internal source of energy of every planet to be from radioactive elements (not ^{26}Al ; that fuel source ran out a long time ago). Assume each planet is, on average, made of chondritic material which today emits $5 \times 10^{-8} \text{ erg s}^{-1} \text{ g}^{-1}$. So given the total mass of each planet, you know the internal luminosity of each planet.

This radiogenic energy is carried conductively through the hard outer lithosphere of each planet. Recall the equation for the conductive heat flux:

$$F = -K\nabla T \tag{30}$$

where K is the thermal conductivity (proportional to, but having wildly different units from, the thermal diffusivity), and T is the temperature. Flux F has units of $\text{erg s}^{-1} \text{ cm}^{-2}$.

At the top of the lithosphere, the temperature is just the surface temperature. You know what that is. (And don't forget that Venus is plenty hotter than you would guess from the blackbody formula.)

At the bottom of the lithosphere (top of the convective mantle), the temperature is $\sim 1200 \text{ K}$, the temperature at which rock starts to become plastic (the viscosity of rock is exponentially sensitive to temperature).

Use all the facts above to estimate the thicknesses of the lithospheres for Venus, Earth, and Mars.

Geologists tells us that “thin” lithospheres are not good for plate tectonics because they are too light and therefore do not subduct (sink) to greater depths (like trying to drown a rubber duck; it just won't go down.) Geologists also tell us that “thick” lithospheres are not good for plate tectonics because they are, well, too thick. Can you begin to understand why the Earth exhibits tectonic motion but the other planets do not?

The flux emitted by each planet equals its internal luminosity divided by the surface area of the planet: $F = \frac{4\pi}{3}\rho R_p^3 \times 5 \times 10^{-8} / 4\pi R_p^2 = \rho R_p \times 5 \times 10^{-8} / 3$, in cgs units. Here R_p is the radius of the planet.

The flux by the heat conduction formula is $F = K(1200 - T_s)/L_{litho}$, where T_s is the surface temperature (in degrees K) and L_{litho} is the thickness of the lithosphere that we wish to solve for. The heat conductivity of rock is $K \approx 4 \times 10^5$ in cgs units. Set our expression for F in this paragraph equal to our expression for F in the previous paragraph and solve for L_{litho} :

$$L_{litho} \approx \frac{3K(1200 - T_s)}{\rho R_p \times 5 \times 10^{-8}} \quad (31)$$

For all planets, I will take $\rho = 3 \text{ g cm}^{-3}$. For Earth, $R_p \approx 6400 \text{ km}$, $T_s \approx 300 \text{ K}$, and $L_{litho,Earth} \approx 113 \text{ km}$. For Venus, $R_p \approx 6000 \text{ km}$, $T_s \approx 700 \text{ K}$, and $L_{litho,Venus} \approx 67 \text{ km}$. For Mars, $R_p \approx 3400 \text{ km}$, $T_s \approx 220 \text{ K}$, and $L_{litho,Mars} \approx 230 \text{ km}$.

Like little bear's porridge, Earth's lithospheric thickness is apparently "just right" for plate tectonics: neither too thin (Venus), nor too thick (Mars).

Problem 3. Sea Floor Spreading

The mid-Atlantic ridge represents an upwelling boundary between two convection cells. New material is flowing away from the ridge at a roughly constant velocity, u .

(a) *Reproduce our order-of-magnitude symbolic expression for u and verify that it is about a few cm/yr.*

If you recall, our derivation of the velocity of the slab was accomplished by setting equal the force of negative buoyancy, which was pulling it down, to the viscous force, which impeded its downward motion. When these two forces are equal the slab moves at a constant velocity u . The force of negative buoyancy is given by the expression:

$$F_{nb} \sim \frac{\delta\rho}{\rho} g \rho V = \frac{\delta\rho}{\rho} g \rho D dx \quad (32)$$

where V is the volume of the slab, D is the length, x is the width, and d is the thickness which increases with distance from the upwelling. The reason the thickness of the slab increases with distance from the upwelling is that heat is conducted away from the slab allowing the top layer to cool and form a crust. Thus, the thickness d will be given by the equation:

$$d \sim \sqrt{\kappa t} \sim \sqrt{\kappa D/u} \quad (33)$$

where κ is the thermal diffusivity of the rock and $t = D/u$ is the time it takes for the slab to travel to the point where it is again subducted.

Next we can make use of the relation $\delta\rho/\rho = \alpha\Delta T$, to rewrite the equation for the force of negative buoyancy. Here α is a coefficient of thermal expansion describing the change in volume of a substance in response to a change in temperature.

$$F_{nb} \sim \alpha\Delta T g \rho D x \sqrt{\kappa D/u} \quad (34)$$

The viscous force resists the downward motion of the slab, and is given by the product of the density, viscosity, a velocity gradient and the contact area of the slab. The contact area is easy to determine, it is simply the surface area Dx . The velocity gradient is trickier, we need to think about the part of the slab where the motion is changing from horizontal to vertical. Hopefully you can convince yourself that the relevant length to go from 0 to u in velocity is D . This means that the viscous force is given by:

$$F_v \sim \rho\nu \frac{u}{D} Dx \sim \rho\nu u x \quad (35)$$

We can now set the negative buoyancy force and the viscous force equal and solve for the velocity u .

$$\alpha\Delta T g \rho D x \sqrt{\kappa D/u} = \rho\nu u x \quad (36)$$

$$\alpha\Delta T g D \sqrt{\kappa D/u} = \nu u \quad (37)$$

$$\frac{\alpha\Delta T g \sqrt{\kappa D^3}}{\nu} = u^{3/2} \quad (38)$$

$$\left[\frac{Ra\kappa}{D^3} \sqrt{\kappa D^3} \right]^{2/3} = u \quad (39)$$

$$(Ra)^{2/3} \frac{\kappa}{D} = u \quad (40)$$

In solving this equation we made use of the relation:

$$Ra = \frac{\alpha\Delta T g D^3}{\kappa\nu} \quad (41)$$

where Ra is the Rayleigh number of the Earth's mantle.

We have shown above that the velocity of the slab is given by $u \sim (Ra)^{2/3} \kappa/D$. We can now evaluate this for the Earth. The Rayleigh number of the convective motions of the mantle is $\sim 10^6$ and the thermal diffusivity of rock is $\kappa \sim 10^{-2}$ in cgs. The length (or depth) of the convective cell is something like 3000 km if you think of the spacing between the mid-ocean ridges and subduction zones on Earth (the size of a tectonic plate). Plugging these numbers in you'll find a speed of $u \sim 3 \times 10^{-7}$ cm/s ~ 10 cm/yr.

(b) Assume Middle-Earth is a planet whose mantle dynamics is very similar to that of the Earth. The One Ring was dropped by Frodo into Mount Doom, where it was swept away into the interior of the planet.

If (contrary to legend) the One Ring can withstand any environmental condition, about how long would it take for the One Ring to re-appear?

As we derived in the previous question, the velocity with which material in a convection cell on Earth moves is about 10 cm/yr. If the Ring were tossed into the mantle of the Earth it would travel along with the convective cell as if it were on a big conveyer belt. Therefore, if it can withstand the fires of Mt. Doom, to estimate how long it will take to reappear, we need to know the length of the conveyer belt. We know this too. The convection cell is roughly a square with sides of length D (the depth is the same as the length, if you recall from our in-class convection demonstration). This means that the distance the Ring travels is about $3D$. The time for the Ring to reappear and mess everything up again in Middle Earth is $t = 3D/u \sim 9 \times 10^7$ years.

(c) The conductive heat flux from the ocean floor decreases with distance away from the ridge. That is, your feet feel hotter if you are standing near the ridge than if you are standing far from the ridge. How does the heat flux, F , scale with distance, x , from the ridge?

A proportionality (scaling relation) is sufficient; there is no need to provide constants.

In fact, such a dependence with distance is measured, and is one of the key pieces of evidence we have that plate tectonics works the way we think it does.

The conductive heat flux is described by the equation $F = -K\nabla T$ where K is the conductivity of rock. If d is the thickness of the crust, the temperature gradient is:

$$\nabla T = \frac{T_{in} - T_{out}}{0 - d} \quad (42)$$

where T_{in} is the temperature at the base of the crust and T_{out} is the temperature at the top of the crust. Think about the case where the crust is the same thickness everywhere. In this case there would be no difference in the conductive heat flux at any distance from the ridge, because the temperature gradient is the same everywhere. In reality the crust gets thicker and thicker the farther from the ridge you go, so the conductive heat flux changes.

In part (a) we figured out how the thickness of the crust changes with time:

$$d \sim \sqrt{\kappa t} \sim \sqrt{\kappa x/u} \quad (43)$$

where $t = x/u$ is the time elapsed since the slab emerged from the ridge and started travelling towards the subduction zone. [Do not confuse the x for this part of the problem with the x is part (a)!] Plugging this into the equation for the conductive flux you can show:

$$F \sim \frac{K\Delta T}{\sqrt{\kappa x/u}} \propto \frac{1}{\sqrt{x}} \quad (44)$$

Problem 4. Helium Rain, or More for Your Money

Saturn's total luminosity is about 1.7 times greater than the power that it absorbs from sunlight. It is thought that the gravitational settling of helium out of the gaseous envelope of Saturn onto the core of Saturn is responsible for this extra power. The pressure in the interior of Saturn is high enough that helium is a liquid. It is a liquid that is immiscible (doesn't mix) with hydrogen, which is also liquid. The liquid helium literally rains to greater depths within the planet, converting its gravitational potential energy to heat.

(a) If all of the helium from Saturn's gaseous envelope rains onto the surface of the rocky core of Saturn, how much energy would be released? Answer in [ergs]. An order-of-magnitude answer is sufficient; you don't need to use any complicated model for Saturn's interior structure. Just remember that Saturn is thought to have a rocky core having a mass of ~ 10 Earth masses, and take Saturn's envelope to be of solar composition (and yes, I really do mean solar composition this time!)

Idealize the planet as a sphere of uniform density ρ , radius $r_s = 6 \times 10^4$ km, and mass m_s , inside of which is a rocky core of density ρ_{core} , radius r_{core} , and mass m_{core} . In our expressions below, we will sometimes make the approximation that $m_s \gg m_{core}$. In actuality, Saturn's mass is about $m_s \approx 100M_\oplus$ and the core mass is about $m_{core} \approx 10M_\oplus$, so this approximation is not bad.

Each spherical shell of helium atoms of mass $4\pi\tilde{r}^2 d\tilde{r}\rho_{He}$ falling from radius \tilde{r} to r_{core} loses a gravitational potential energy equal to

$$- \int_{r=\tilde{r}}^{r=r_{core}} \frac{G(4\pi\tilde{r}^2 d\tilde{r}\rho_{He})[m_{core} + 4\pi\rho(r^3 - r_{core}^3)/3]}{r^2} dr \quad (45)$$

where the term in the square brackets equals the mass enclosed within radius r . Integrate over all shells of helium to obtain the total gravitational energy release:

$$+ \int_{\tilde{r}=r_s}^{\tilde{r}=r_{core}} \int_{r=\tilde{r}}^{r=r_{core}} \frac{G(4\pi\tilde{r}^2 d\tilde{r}\rho_{He})[m_{core} + 4\pi\rho(r^3 - r_{core}^3)/3]}{r^2} dr \quad (46)$$

Do the r -integral first and then the \tilde{r} -integral. The integrals are elementary and yield, APPROXIMATELY,

$$E \sim \frac{3}{2} \frac{Gm_s^2}{r_s} \frac{\rho_{He}}{\rho} \left(\frac{1}{5} - \frac{r_{core}^2}{3r_s^2} + \frac{2r_{core}^5}{15r_s^5} \right) + 3 \frac{\rho_{He}}{\rho} \frac{m_s m_{core}}{r_s} \left(\frac{r_s}{3r_{core}} - \frac{1}{2} + \frac{r_{core}^2}{6r_s^2} \right) \quad (47)$$

If we assume that $\rho_{core} \approx 7 \text{ g/cm}^3$ and $m_{core} = 10M_\oplus$, then $r_{core} \approx 1.3 \times 10^9 \text{ cm}$ and $r_{core}/r_s \approx 0.2$. For solar composition gas, $\rho_{He}/\rho = 0.25$. Then $E \sim 3 \times 10^{41} \text{ erg}$.

(b) *How much excess energy is released by Saturn over the age of the solar system, assuming its luminosity has been constant for this time?*

Saturn radiates 1.7 times as much as it absorbs. The excess luminosity is therefore 0.7 times what it absorbs. It absorbs a power $F(1 - A)\pi r_s^2$, where $F = L_\odot/4\pi(10 \text{ AU})^2$, and the bond albedo $A = 0.34$. Then the excess energy radiated by Saturn over a time t is $E_x \sim 1 \times 10^{41} \text{ erg}$.

(c) *Compare (a) to (b) and comment.*

$E_x \sim E$. Considering that we made a drastic approximation of uniform density, the equality has no right to be as close as it is, but the fact that it is close makes helium rain an excellent candidate for explaining the extra power radiated by Saturn.