

Astro 162 – Planetary Astrophysics – Problem Set 9

Due Thursday in class.

Readings: Course Reader, entire chapter by Mark Marley on “Interiors of the Giant Planets.”

Problem 1. The Solar Core

Is the core of the Sun degenerate (or is the pressure there instead given by the ideal gas law)? Justify your answer quantitatively.

Problem 2. Liquid Giants

This problem is adapted from problem 11.2 of Landstreet. Mostly I have exchanged his SI units for more sensible units. You can read the wording of his problem for hints, if you need them.

In Saturn, the temperature is about 110 K at a pressure of 0.5 bar (1 bar = 10^6 dyne cm^{-2}). Below this level the atmosphere is convecting and the temperature increases with depth s at the quasi-adiabatic rate of $dT/ds \approx 0.7 \text{ K km}^{-1}$. Take the atmosphere to be composed purely of molecular hydrogen.

Gas liquifies once its density approaches that of liquid, of order $\sim 1 \text{ g cm}^{-3}$. ESTIMATE the depth, s , at which molecular hydrogen liquifies. Use hydrostatic equilibrium, and assume that the gravitational acceleration g is constant (it will be, roughly, as long as you don't wander too close to the center of the planet).

Thereby argue that the gas giants, Jupiter and Saturn, are more appropriately called “liquid giants.”

Problem 3. The Incredible Invariant Radius

(a) Combine the equation for hydrostatic equilibrium,

$$\frac{dP}{dr} = g\rho, \quad (1)$$

Poisson's equation for gravity,

$$\nabla^2\phi = 4\pi G\rho, \quad (2)$$

and an approximate equation of state for liquid hydrogen,

$$P = K\rho^2, \tag{3}$$

to derive a single second-order differential equation for density as a function of radius, $\rho(r)$. Here ϕ is the gravitational potential, $g = -\nabla\phi$ is the gravitational acceleration, P is pressure, and $K = 2.7 \times 10^{12}$ [cgs] is a constant appropriate for liquid hydrogen. Assume spherical symmetry (*so work in spherical coordinates*). Notice that I am asking you to use $P \propto \rho^2$, which is close, but not quite equal to what we derived in class, $P \propto \rho^{5/3}$. The latter law is really only appropriate for a pure degenerate gas of free electrons, and masses of order a Jupiter only achieve pressure ionization over a fraction of their interiors, not throughout.

(b) The solution to the equation you have derived (assuming you have the right one; at least check your units) is

$$\frac{\rho}{\rho_0} = \frac{\sin(\pi r/R)}{(\pi r/R)} \tag{4}$$

Note that R is the outer radius of the body, since ρ vanishes at $r = R$. Derive, using (a) and the equation above, a symbolic expression for R .

(c) Numerically evaluate R for Saturn, Jupiter, and a brown dwarf having 10 Jupiter masses.

Problem 4. The Nougat in the Center

The leading theory for how gas giants form is by “core nucleation.”

Imagine a body of rock and ice orbiting within the primordial solar nebula. If the mass of the body exceeds a certain critical threshold, gas from the solar nebula will accrete onto the core at a *runaway* pace (see figure 12.15 in the Course Reader), thereby forming a gas giant like Jupiter. The “critical core mass” is widely cited to be between 5–10 M_{\oplus} .

This problem tries to understand why the critical core mass is between 5–10 M_{\oplus} .

Consider a rock/ice core of mass M , radius R , and internal density ρ_p sitting in an infinite, homogeneous sea of gas (i.e., the solar nebula) whose density, pressure, and temperature at *infinity* are ρ_0 , P_0 , and T_0 , respectively. The core gravitationally attracts gas onto itself. This accreted gas forms a bound, hydrostatic atmosphere. The gas density is naturally highest at the rocky core’s surface, and grades into the ambient density ρ_0 at infinite distance from the core.

Assume the (ideal) gas behaves adiabatically: $P = P_0(\rho/\rho_0)^\gamma$, where you can take $\gamma = 7/5$ as befits a classically excited, rigid diatomic rotor like molecular hydrogen.

Also the mean molecular weight of the gas is μ .

(a) Show, to order-of-magnitude, that the density of the gas at the core's surface is

$$\rho_s \sim \left(\frac{2GM\mu}{7kT_0R} \right)^{5/2} \rho_0. \quad (5)$$

To derive this result, use hydrostatic equilibrium with the appropriate boundary conditions at infinity. The only approximation you will have to make is $\rho_s \gg \rho_0$ (the density of gas at the core's surface is much larger than the density of gas at infinity; this is a fine approximation).

(b) Let's estimate the mass in the atmospheric envelope bound to the core as

$$M_{\text{env}} \sim 4\pi R^2 \rho_s H, \quad (6)$$

where $H \sim kT_s/(\mu g)$ is the gas scale height at the core's surface, T_s is the gas temperature at the surface of the core, and $g \sim GM/R^2$ is the gravitational acceleration at the core's surface. To solve for T_s , recall our assumption that the gas behaves adiabatically.

Solve for the critical radius of the core, R_c , such that $M_{\text{env}} = M$. Give an analytic expression for R_c in terms of the variables given.

Also give a numerical estimate for the critical mass, M_c , in terms of M_\oplus . See how close you come to what the professionals get. For your numerical estimate, assume that $\rho_p \sim 6 \text{ g cm}^{-3}$, $\mu = 3 \times 10^{-24} \text{ g}$, and take T_0 , ρ_0 , and P_0 to be appropriate for the minimum-mass solar nebula at Jupiter's heliocentric distance of 5 AU (these should be derivable from your notes from a few months ago).

Finally, comment on the physical significance of this derivation. Why, for example, is the condition $M_{\text{env}} = M$ so special? And do we know whether the cores of Jupiter and Saturn indeed have such critical masses?