

Astro 162 – Planetary Astrophysics – Solutions to Set 9

Problem 1. *The Solar Core*

Is the core of the Sun degenerate (or is the pressure there instead given by the ideal gas law)? Justify your answer quantitatively.

We can either look up the core temperature and core number density of the Sun (completely valid decision for this problem), or we can try to estimate our way there. Let's try the latter approach first. I remember hydrogen fusion kicks in at about $T \sim 10^7$ K, so I will use that as the central temperature. To estimate the number density, one might just say ρ/μ , where $\rho \sim M_\odot/V_\odot$, $\mu \sim 2 \times 10^{-24}$ g, and V_\odot is the volume of the Sun. This naive procedure gives $\rho \sim 1 \text{ g cm}^{-3}$. Sadly, I also remember that this is too low an estimate; the central density is actually a few orders of magnitude higher than the mean density.

To estimate the true core density, I will employ the following ruse. I remember from taking Astro-7-esque classes in the past that I can estimate the central pressure pretty accurately by a brazen use of hydrostatic equilibrium:

$$\frac{P}{R} \sim g\rho \sim \frac{GM}{R^2} \frac{M}{R^3}. \quad (1)$$

Therefore

$$P \sim \frac{GM^2}{R^4}. \quad (2)$$

Let's assume that the core gas is ideal and not degenerate. Now I understand the whole point of the problem is to decide this issue; but we can assume it's ideal and then check to see if our assumption is correct, or at least close to being correct. Then $P = \rho kT/\mu$, and therefore

$$\frac{\rho}{\mu} \sim \frac{GM^2}{R^4 kT} \quad (3)$$

where we have plugged in for P above. Plugging in $M = M_\odot$, $R = R_\odot$, and $T = 10^7$ K, I get

$$n_p \sim \frac{\rho}{\mu} \sim 8 \times 10^{24} \text{ cm}^{-3} = n_e \quad (4)$$

where n_p and n_e are the number densities of protons and of electrons, respectively.

Our estimates above are not too far off from the truth, where the truth is given by detailed calculations of stellar structure:

$$T \sim 1.6 \times 10^7 \text{ K} \quad (5)$$

$$n_p = n_e \sim 8 \times 10^{25} \text{ cm}^{-3} \quad (6)$$

This n_p implies that $\rho \sim 160 \text{ g cm}^{-3}$. This central density for the Sun is probably worth remembering.

Now we finally check to see whether such a temperature and density corresponds to a degenerate gas. First we check to see whether the electrons are degenerate. Calculate the de Broglie wavelength of each electron,

$$\lambda_e = \frac{h}{p_e} \quad (7)$$

where the momentum of the electron, p_e , is given by

$$\frac{p^2}{2m_e} \sim \frac{3}{2}kT. \quad (8)$$

Then

$$\lambda_e = \frac{h}{\sqrt{3kTm_e}} \sim 2 \times 10^{-9} \text{ cm}. \quad (9)$$

Compare this to the mean spacing between electrons,

$$n_e^{-1/3} \sim 2 \times 10^{-9} \text{ cm}. \quad (10)$$

The fact that $\lambda_e \sim n_e^{-1/3}$ implies that the electron gas is actually mildly degenerate. So you shouldn't think of degeneracy as only afflicting "compact objects" like white dwarfs, brown dwarfs, and neutron stars. The electrons in the core of the Sun are pretty tightly packed. Note that had we used our crude estimates, we would have arrived at pretty much the same answer (the $1/3$ power in $n_e^{-1/3}$ saves us).

What about the protons? We can substitute m_p for m_e everywhere above to find that $\lambda_p \ll n_p^{-1/3}$ and therefore the proton gas is not degenerate (i.e., the proton gas behaves like an ideal gas).

Problem 2. *Liquid Giants*

This problem is adapted from problem 11.2 of Landstreet. Mostly I have exchanged his SI units for more sensible units. You can read the wording of his problem for hints, if you need them.

In Saturn, the temperature is about 110 K at a pressure of 0.5 bar (1 bar = 10^6 dyne cm^{-2}). Below this level the atmosphere is convecting and the temperature increases with depth s at the quasi-adiabatic rate of $dT/ds \approx 0.7 \text{ K km}^{-1}$. Take the atmosphere to be composed purely of molecular hydrogen.

Gas liquifies once its density approaches that of liquid, of order $\sim 1 \text{ g cm}^{-3}$. Liquification also requires that the temperature be just right, but let us ignore the temperature dependence for this problem and just focus on the density dependence.

ESTIMATE the depth, s , at which molecular hydrogen liquifies. Use hydrostatic equilibrium, and assume that the gravitational acceleration g is constant (it will be, roughly, as long as you don't wander too close to the center of the planet).

Thereby argue that the gas giants, Jupiter and Saturn, are more appropriately called "liquid giants."

Hydrostatic equilibrium states

$$\frac{dP}{ds} = g\rho \tag{11}$$

where s increases downwards, and g is assumed to be a positive constant. Insert the ideal gas law into the left-hand-side to write

$$\frac{dP}{ds} = \frac{k}{\mu} \left(T \frac{d\rho}{ds} + \rho \frac{dT}{ds} \right). \tag{12}$$

Now T is a known function based on the information given in the problem:

$$T = T_0 + \frac{dT}{ds}s \tag{13}$$

where $T_0 = 110 \text{ K}$ is the temperature at the reference level $s = 0$ (where the pressure, $P_0 = 0.5 \text{ bar}$). Let's write this more conveniently as

$$T = T_0 + \beta s \quad (14)$$

where $\beta \equiv dT/ds = 0.7 \text{ K km}^{-1} = 7 \times 10^{-6} \text{ (cgs)}$ is a known constant. Plug everything into (12):

$$\frac{dP}{ds} = \frac{k}{\mu} (T_0 + \beta s) \frac{d\rho}{ds} + \frac{k}{\mu} \beta \rho \quad (15)$$

Insert into (11) and re-arrange:

$$(T_0 + \beta s) \frac{d\rho}{ds} = \left(\frac{\mu g}{k} - \beta \right) \rho \quad (16)$$

This is a separable equation,

$$\frac{d\rho}{\rho} = \left(\frac{\mu g}{k} - \beta \right) \frac{ds}{T_0 + \beta s}, \quad (17)$$

whose solution is

$$\rho = C (T_0 + \beta s)^{\frac{\mu g}{k\beta} - 1} \quad (18)$$

(after exponentiating both sides). Solve for the constant of integration C using the boundary condition at $s = 0$. At $s = 0$, $\rho = \rho_0 = \mu P_0 / (kT_0) = 1.1 \times 10^{-4} \text{ g cm}^{-3}$, where $\mu = 3.4 \times 10^{-24} \text{ g}$. The exponent in equation (18) equals

$$\frac{\mu g}{k\beta} - 1 \approx 2.5 \quad (19)$$

where $g = GM_S/R_S^2 = 10^3 \text{ [cgs]}$ is the gravitational acceleration near the Saturnian surface (coincidentally nearly the same as that on Earth). Therefore, putting it all together,

$$C \approx \frac{\rho_0}{T_0^{2.5}} \approx 8.7 \times 10^{-10} \text{ [cgs]} \quad (20)$$

Now all the constants are known. Finally, use (18) to solve for the depth s at which $\rho = \rho_{\text{liquid}} \sim 1 \text{ g cm}^{-3}$:

$$s_{\text{liquid}} \approx \frac{1}{\beta} \left[\left(\frac{\rho_{\text{liquid}}}{C} \right)^{\frac{1}{2.5}} - T_0 \right] \approx 6000 \text{ km} \quad (21)$$

Thus, we need to travel about 6000 km downwards to attain liquid conditions. Since the radius of Saturn is about 60000 km, we can conclude (modulo the dependence of the phase diagram on temperature) that most of Saturn is liquid.

Problem 3. *The Incredible Invariant Radius*

(a) *Combine the equation for hydrostatic equilibrium,*

$$\frac{dP}{dr} = g\rho, \quad (22)$$

Poisson's equation for gravity,

$$\nabla^2 \phi = 4\pi G\rho, \quad (23)$$

and an approximate equation of state for liquid hydrogen,

$$P = K\rho^2, \quad (24)$$

to derive a single second-order differential equation for density as a function of radius, $\rho(r)$. Here ϕ is the gravitational potential, $g = -\nabla\phi$ is the gravitational acceleration, P is pressure, and $K = 2.7 \times 10^{12}$ [cgs] is a constant appropriate for liquid hydrogen. Assume spherical symmetry (so work in spherical coordinates). Notice that I am asking you to use $P \propto \rho^2$, which is close, but not quite equal to what we derived in class, $P \propto \rho^{5/3}$. The latter law is really only appropriate for a pure degenerate gas of free electrons, and masses of order a Jupiter only achieve pressure ionization over a fraction of their interiors, not throughout.

Note further that in the equation for hydrostatic equilibrium, the right-hand-side has no minus sign, because the g for this problem is negative already (gravity points inward).

First insert the equation of state into the condition for hydrostatic equilibrium to find

$$\frac{dP}{dr} = 2K\rho \frac{d\rho}{dr} = -\rho \nabla\phi = -\rho \frac{d\phi}{dr} \quad (25)$$

where for the second-to-last equality we have also used $g = -\nabla\phi$.

Now expand Poisson's equation, remembering the formula for ∇^2 in spherical coordinates for a spherically symmetric body:

$$4\pi G\rho = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) \quad (26)$$

Finally, insert (25) for $d\phi/dr = -2Kd\rho/dr$ into (26), expand, and re-arrange:

$$4\pi G\rho = -\frac{1}{r^2} \frac{d}{dr} \left(r^2 2K \frac{d\rho}{dr} \right) \quad (27)$$

$$= -\frac{2K}{r^2} \left(2r \frac{d\rho}{dr} + r^2 \frac{d^2\rho}{dr^2} \right) \quad (28)$$

$$\rightarrow \boxed{\frac{d^2\rho}{dr^2} + \frac{2}{r} \frac{d\rho}{dr} + \frac{2\pi G}{K} \rho = 0} \quad (29)$$

(b) *The solution to the equation you have derived (assuming you have the right one; at least check your units) is*

$$\frac{\rho}{\rho_0} = \frac{\sin(\pi r/R)}{(\pi r/R)} \quad (30)$$

Note that R is the outer radius of the body, since ρ vanishes at $r = R$. Derive, using (a) and the equation above, a symbolic expression for R .

Plug the solution into our differential equation in (a). After a lot of uninteresting algebra, we find that

$$\boxed{R = \sqrt{\pi K/2G}} \quad (31)$$

(c) *Numerically evaluate R for Saturn, Jupiter, and a brown dwarf having 50 Jupiter masses.*

Well, our expression for R is independent of mass. For all bodies, $\boxed{R \approx 8 \times 10^9 \text{ cm}}$ —which is actually not far from the truth! (Note that this problem permits some crude insight into why Saturn is “anomalously” low in density—Saturn has 1/3 the mass of Jupiter, but its size and therefore its volume is about the same as that of Jupiter. Thus, Saturn's density should be lower than that of Jupiter.)

Problem 4. *The Nougat in the Center*

The leading theory for how gas giants form is by “core nucleation.”

Imagine a body of rock and ice orbiting within the primordial solar nebula. If the mass of the body exceeds a certain critical threshold, gas from the solar nebula will accrete onto the core at a runaway pace (see figure 12.15 in the Course Reader), thereby forming a gas giant like Jupiter. The “critical core mass” is widely cited to be between 5–10 M_{\oplus} .

This problem tries to understand why the critical core mass is between 5–10 M_{\oplus} .

Consider a rock/ice core of mass M , radius R , and internal density ρ_p sitting in an infinite, homogeneous sea of gas (i.e., the solar nebula) whose density, pressure, and temperature at infinity are ρ_0 , P_0 , and T_0 , respectively. The core gravitationally attracts gas onto itself. This accreted gas forms a bound, hydrostatic atmosphere. The gas density is naturally highest at the rocky core’s surface, and grades into the ambient density ρ_0 at infinite distance from the core.

Assume the (ideal) gas behaves adiabatically: $P = P_0(\rho/\rho_0)^\gamma$, where you can take $\gamma = 7/5$ as befits a classically excited, rigid diatomic rotor like molecular hydrogen. Also the mean molecular weight of the gas is μ .

(a) Show, to order-of-magnitude, that the density of the gas at the core’s surface is

$$\rho_s \sim \left(\frac{2GM\mu}{7kT_0R} \right)^{5/2} \rho_0. \quad (32)$$

To derive this result, use hydrostatic equilibrium with the appropriate boundary conditions at infinity. The only approximation you will have to make is $\rho_s \gg \rho_0$ (the density of gas at the core’s surface is much larger than the density of gas at infinity; this is a fine approximation).

Once again, perhaps the most-often used equation in all of astrophysics states:

$$\frac{dP}{dr} = -\frac{GM}{r^2}\rho \quad (33)$$

where $r > R$ measures the distance from the center of the core. Substitute the given adiabatic relation for P to find

$$\frac{7}{5} \frac{P_0}{\rho_0^{7/5}} \rho^{2/5} \frac{d\rho}{dr} = -\frac{GM}{r^2} \rho \quad (34)$$

This is separable:

$$\frac{7}{5} \frac{P_0}{\rho_0^{7/5}} \rho^{-3/5} d\rho = -\frac{GM}{r^2} dr \quad (35)$$

and the solution is

$$\frac{7}{2} \frac{P_0}{\rho_0^{7/5}} \rho^{2/5} = \frac{GM}{r} + C. \quad (36)$$

To solve for C , use the boundary condition at $r = \infty$ that $\rho = \rho_0$:

$$C = \frac{7}{2} \frac{P_0}{\rho_0}. \quad (37)$$

Plug the constant C back into (36):

$$\frac{7}{2} \frac{P_0}{\rho_0^{7/5}} (\rho^{2/5} - \rho_0^{2/5}) = \frac{GM}{r} \quad (38)$$

Now we want $\rho = \rho_s$ at $r = R$ (the surface). Assume that $\rho = \rho_s \gg \rho_0$ to drop the $\rho_0^{2/5}$ term in the above equation:

$$\frac{7}{2} \frac{P_0}{\rho_0^{7/5}} \rho_s^{2/5} \sim \frac{GM}{R} \quad (39)$$

and solve for ρ_s :

$$\rho_s \sim \left(\frac{2}{7}\right)^{5/2} \left(\frac{GM}{R}\right)^{5/2} \left(\frac{\rho_0}{P_0}\right)^{5/2} \rho_0. \quad (40)$$

Get rid of P_0 using the ideal gas law to find the desired relation:

$$\boxed{\rho_s \sim \left(\frac{2GM\mu}{7kT_0R}\right)^{5/2} \rho_0.} \quad (41)$$

(b) Let's estimate the mass in the atmospheric envelope bound to the core as

$$M_{\text{env}} \sim 4\pi R^2 \rho_s H, \quad (42)$$

where $H \sim kT_s/(\mu g)$ is the gas scale height at the core's surface, T_s is the gas temperature at the surface of the core, and $g \sim GM/R^2$ is the gravitational acceleration at the core's surface. To solve for T_s , recall our assumption that the gas behaves adiabatically.

Solve for the critical radius of the core, R_c , such that $M_{\text{env}} = M$. Give an analytic expression for R_c in terms of the variables given.

Also give a numerical estimate for the critical mass, M_c , in terms of M_{\oplus} . See how close you come to what the professionals get. For your numerical estimate, assume that $\rho_p \sim 6 \text{ g cm}^{-3}$, $\mu = 3 \times 10^{-24} \text{ g}$, and take T_0 , ρ_0 , and P_0 to be appropriate for the minimum-mass solar nebula at Jupiter's heliocentric distance of 5 AU (these should be derivable from your notes from a few months ago).

Finally, comment on the physical significance of this derivation. Why, for example, is the condition $M_{\text{env}} = M$ so special? And do we know whether the cores of Jupiter and Saturn indeed have such critical masses?

To know H , we need to know T_s . But T_s is related to T_0 by the adiabatic relation (for $\gamma = 7/5$):

$$T_s = T_0 \left(\frac{\rho_s}{\rho_0} \right)^{2/5}. \quad (43)$$

Now it is just a question of plugging in everything into M_{env} :

$$M_{\text{env}} \sim 4\pi R^2 \left(\frac{2GM\mu}{7kT_0R} \right)^{5/2} \rho_0 \frac{kR^2T_0}{GM\mu} \frac{2GM\mu}{7kT_0R} \quad (44)$$

$$\sim \frac{8\pi}{7} R^2 \left(\frac{2GM\mu}{7kT_0R} \right)^{5/2} \rho_0 R \quad (45)$$

$$\sim \frac{8\pi}{7} \rho_0 R^8 \left(\frac{8\pi G\mu\rho_p}{21kT_0} \right)^{5/2}. \quad (46)$$

Now set M_{env} equal to $M = (4\pi/3)\rho_p R^3$ and solve for $R = R_c$:

$$\boxed{R_c = \left(\frac{7\rho_p}{6\rho_0} \right)^{1/5} \left(\frac{21kT_0}{8\pi G\mu\rho_p} \right)^{1/2}}. \quad (47)$$

Now at a heliocentric distance of 5 AU, $\rho_0 \sim 10^{-11} \text{ g cm}^{-3}$ and $T_0 \sim 75 \text{ K}$, as you can readily verify from your notes on the minimum-mass solar nebula. Plugging into the above equation, we get $R_c \sim 2 \times 10^{10} \text{ cm} \sim 2 \times 10^5 \text{ km}$. Then it immediately follows that

$M_c \sim 2 \times 10^{32} \text{ g} \sim 3 \times 10^4 M_{\oplus}$, which sadly is nowhere close to what the professionals get! Stay tuned to next week's problem set.

The significance of setting $M_{\text{env}} = M$ is that for $M > M_c$, the mass growth grows super-exponentially fast with time, as described in lecture. For $M < M_c$, growth is quasi-linear. As far as the observations go: there is some positive evidence that Saturn has a rocky core, as the plot in the Course Reader indicates. But no such evidence exists for Jupiter. The error bars are large enough that both planets might not have rocky cores at all.