

Homework #1

1. The density of the interior of the sun is significantly larger than that of water. a) Provide a quantitative relation between the temperature and density of a star which indicates when we can treat it as a gas throughout its interior, in spite of the very high densities. Is our assumption valid at the center of the sun? Note that your answer should only involve classical physics, not QM. b) If all stars have roughly the same central temperature (that of the sun), use a scaling argument to determine the stellar mass at the which the simple non-interacting ideal gas assumption breaks down.
2. In class I derived the relationship between the luminosity and mass of stars using radiative diffusion. We will carry out many related scaling arguments during this course.
 - a) To familiarize yourself with the process, derive the relationship between the luminosity, mass and radius of stars assuming radiation transport of energy and that the opacity is given by free-free absorption, $\kappa \approx 10^{23} \rho T^{-7/2} \text{ cm}^2 \text{ g}^{-1}$.
 - b) If all stars have roughly the same central temperature and are supported by gas pressure, derive the resulting mass-luminosity relationship for stars.
 Note that for a) and b) you only need to give a proportionality between L , M , and R (a) or L and M (b). You don't need to keep track of the constants in the various equations.
 - c) Give a quantitative argument as to whether free-free opacity dominates electron scattering opacity in high mass stars or low mass stars.

3. *Polytropes*

- a) The mass M of a star is given by

$$M = \int_0^R 4\pi r^2 \rho(r) dr$$

Use the Lane-Emden equation for polytropes to rewrite this in terms of the central density as

$$\rho_c = \bar{\rho} a_n = \left(\frac{3M}{4\pi R^3} \right) a_n$$

where a_n , the ratio of the central density to the mean density, is a function you should determine that depends only on the solution to the Lane-Emden equation (you cannot evaluate a_n in general without numerically solving for $\theta[\xi]$).

- b) Show that the central pressure of a polytrope can be written as

$$P_c = \frac{4\pi G \rho_c^2 a^2}{n+1}$$

where $a \neq a_n$ is the constant (with units of length) defined in lecture (note that the polytropic relation $P = K\rho^\gamma$ can be used to write $K = P_c/\rho_c^\gamma$). Use this result and a) to derive an expression for the central pressure of a polytropic model of the form

$$P_c = \left(\frac{GM^2}{R^4} \right) c_n$$

Also show that the central pressure of a polytrope can be written as

$$P_c = d_n GM^{2/3} \rho_c^{4/3} \tag{1}$$

where d_n depends on a_n and c_n .

c) Integrate the Lane-Emden equation numerically for $n = 3$ and $n = 1.5$ polytropes. To do this, start from (dimensionless radius) $\xi = 0$ and integrate outwards until the dimensionless density $\theta = 0$ (this is the surface of the star). What are the values of a_n , c_n , and d_n for $n = 1.5$ and $n = 3$? We will use the $n = 1.5$ values extensively in our discussion of fully convective stars. Which polytrope is more centrally concentrated? If the sun is roughly an $n = 3$ polytrope (as in Eddington's standard model), what is the central density and temperature of the sun (assume gas pressure dominated and $\mu = 0.6$)? Compare these values to the correct answer. Note how the use of a simple polytropic model allows us to improve on our order of magnitude estimates of interior stellar quantities by providing a reasonable, although not exact, run of density and temperature.

An easy way to check that your numerical calculation is working correctly is to compare your numerical results with the analytical solutions for $n = 0$, $n = 1$, or $n = 5$.

d) Let's use these results for something useful. Low mass stars are fully convective and are thus well described by $n = 1.5$ polytropes (for reasons we will see). Assume that low mass stars have a mass radius relation given by $R \propto M$ (reasonably accurate) and that $\langle \rho \rangle = 1 \text{ g cm}^{-3}$ for $M = 1M_\odot$. Now show that there is a critical stellar mass below which electron degeneracy pressure at the center of the star becomes comparable to the total pressure implied by hydrostatic equilibrium, in which case our simple assumption of gas pressure support breaks down. As will discuss later, this result is crucial for understanding the minimum mass of stars.¹

¹There is one subtlety worth noting about this problem. If you assume that the polytropic constant K is the same for stars of any mass, then that uniquely implies $R \propto M^{-1/3}$ for an $n = 3/2$ polytrope. Our assumption here, that $R \propto M$, is equivalent to assuming a constant central temperature if gas pressure dominates, and corresponds to a polytropic constant K that depends on the mass of the star. A constant central temperature is not unreasonable, as we shall see, because of the properties of energy generation by fusion.