Homework # 3[1]

1. Small amounts of Deuterium are made in the Big Bang. D is destroyed in the interiors of stars via the reaction $p + D \rightarrow {}^3He + \gamma$. The S value for D-burning is 2.5×10^{-4} keV-barn, each reaction releases ≈ 5.5 MeV, and the cosmic abundance of D from the Big Bang is $n_D \approx 2 \times 10^{-5} n_H$. Let's focus on a low mass fully convective star undergoing KH contraction; such a star can be reasonably well modeled as an $n = 3/2$ polytrope.

   a) What is the Gamow energy for D fusion? Write down the resulting thermally averaged cross-section for D fusion.

   b) In class we derived a quantitative model for the Kelvin-Helmholtz contraction of a low mass star as it approaches the main sequence. Use these results to calculate the local contraction time $t_c \equiv R/|dR/dt|$ as a function of the mass and radius of the star. Does the contraction time get shorter or longer as the star contracts?

   c) What is the lifetime t_D of a D nucleus at the center of the star in terms of the local density and temperature? Use the properties of $n = 3/2$ polytropes to write t_D as a function of M and R. Does the D lifetime get shorter or longer as the star contracts?

   d) For any mass M show that there is a critical radius R_D at which $t_D = t_c$. This represents the radius (time) at which D starts to undergo significant fusion. Give the numerical value of R_D for $M = 0.03$ and $0.1\,M_\odot$. For each of these two cases, also determine the central temperature of the star T_c and the D lifetime t_D when $R = R_D$. Does D fusion occur before or after the star reaches the main sequence?

   e) Explain quantitatively whether D fusion can halt (at least temporarily) the KH contraction of the star. If so, how long does the "D main sequence" last for the two cases considered in part d) above?

   Now that you have finished this problem you might find it interesting to look at some of the figures in Burrows et al., 2001, RVMP, 73, 719 which show evolutionary calculations of the contraction of low mass stars and brown dwarfs and the effects of D fusion (I showed some of these in class as well). After D fusion comes Li fusion and then, if M is big enough, H fusion.

2. In this problem we will determine the main sequence for fully convective low mass stars. We showed in lecture that fully convective stars have $T_{eff} \approx 2600(L/L_\odot)^{1/102}(M/M_\odot)^{7/51}$ K. As also noted in lecture, the coefficient here is a little low and should be closer to 4000 K. In this case, we can also write this result as $L \approx 0.2(M/M_\odot)^{4/7}(R/R_\odot)^{102/49}\,L_\odot$. Let's approximate this as $L \approx 0.2(M/M_\odot)^{4/7}(R/R_\odot)^2\,L_\odot$ to get rid of the one irritating exponent. Call this luminosity L_{conv} since it is derived from the properties of energy transport alone (convective interior + radiative atmosphere with H^- opacity). The luminosity of a star is also given by

$$L_{fusion} = \int 4\pi r^2 \rho \epsilon(T, \rho) dr \qquad (1)$$

   where ϵ is due to the proton-proton chain for low mass stars (given in lecture or HK). The main sequence is determined by $L_{conv} = L_{fusion}$.

   a) Use scaling arguments to derive the power-law relations $R(M)$, $L(M)$, $T_c(M)$ and $L(T_{eff})$ for fully convective stars. Approximate $\epsilon \propto \rho T^\beta$ with an appropriate choice of β. Note that it is not reasonable to estimate the properties of low mass stars by scaling from the properties of the sun, since the sun is not a fully convective star. Instead we need to actually determine the structure of some fully convective star. This is what we will do in the rest of the problem.

[1]This is a long homework. Start early.

We can significantly improve on the above scaling arguments by using the fact that fully convective stars are n = 3/2 polytropes. It turns out that for a polytrope, ϵ in equation (1) can be Taylor expanded near the center to yield

$$L_{fusion} \simeq \frac{2.4\,\epsilon_c\,M}{(3+\beta)^{3/2}} \tag{2}$$

where I have again approximated $\epsilon \propto \rho T^\beta$ and where ϵ_c is ϵ evaluated at the center of the star. I am not asking you prove equation (2). You will have to trust me. Note that for a typical value of β for the pp chain, equation (2) says that $L_{fusion} \simeq 0.1\epsilon_c M$. This makes sense because fusion only takes place at the center of the star (not all of the mass participates).

b) Use the results for n = 3/2 polytropes from HW 1 and in class to write the central temperature of the star T_c, central density ρ_c, and pp energy generation at the center of the star ϵ_c in terms of the mass M and radius R. Assume $X = 0.7$ and $\mu = 0.6$ (typical for stars just reaching the main sequence). Note that you should give expressions for T_c, ρ_c, and ϵ_c here, with constants and real units, not just scaling relationships. So that the constants in front of your expressions are reasonable, please normalize M to $M_\odot$ and R to $R_\odot$.

c) Use equation (2), the results of b), and $L_{conv} = L_{fusion}$ on the main sequence to determine the $R(M)$, $L(M)$, $T_c(M)$, and $L(T_{eff})$ relations for fully convective stars. If you use the same β, your expressions here should be the same as in a) *except that you should now be able to determine the absolute normalization for $R(M)$, $L(M)$, etc., i.e., you have determined the true luminosity and radius of a fully convective star from first principles.* In doing this problem, remember that β is temperature dependent so make sure you check that your value of β is reasonable given the resulting central temperature that you calculate.[2]

d) What are your predicted luminosities, radii, and effective temperatures for main sequence stars with $M = 0.1$ and $0.3M_\odot$? Compare your values to the values of $L = 0.01L_\odot$, $R = 0.3R_\odot$, and $T_{eff} = 3450$ K for $M = 0.3M_\odot$ and $L = 10^{-3}L_\odot$, $R = 0.11R_\odot$, and $T_{eff} = 3000$ K for $M = 0.1M_\odot$ from HK.

3. In the file available on the course website, I give the results of calculations of the hydrogen burning main sequence using a simple stellar structure code incorporating analytic opacities, nuclear energy generation rates etc. The columns are hydrogen mass fraction, metal mass fraction, log(M/$M_\odot$), log($T_{\rm eff}$ [K]), log(L/$L_\odot$), log(T_c [K]), log(ρ_c [g cm^{-3}]), and log(R/$R_\odot$). This code is reasonably accurate relative to much more complicated calculations. The file contains solutions for the full (zero-age) main sequence with CNO and p-p fusion and also results for the high mass ($M \geq M_\odot$) main sequence with CNO fusion turned off and the low mass ($M \leq M_\odot$) main sequence with no p-p fusion.

   a) For the low mass main sequence with no p-p fusion, use scaling arguments to explain why the i) central temperatures, ii) luminosities, and iii) radii of stars change the way they do (or don't, as the case may be) relative to the full main sequence with p-p fusion. You do not need to quantitatively explain the exact scalings found in the numerical calculation. Instead, I am just looking for a physical/semi-quantitative explanation of the differences (or lack there of) between when pp fusion is present and when it is not included.

   b) Do the same for the high mass main sequence without CNO fusion.

[2] The right way to do this is to do an iteration where you guess a β, find T_c, adjust β to be appropriate for the new T_c, re-calculate T_c, etc. You don't have to do something that fancy. But do pick a reasonable β appropriate for cooler low mass stars, not one that is appropriate at 10^8 K!

4. In this problem we will quantify some properties of the CNO cycle. Some of this material is discussed in Clayton, and some is not. Show all of your work.

   a). Given the thermally averaged cross section derived in class, write down the general expression for the lifetime of a nucleus with charge Z and atomic number A reacting with H.

   b) The CNO cycle involves 4 proton captures and two beta decays. Oxygen only undergoes beta decay while C and N (two isotopes each) undergo proton captures. As the file from the previous problem shows, high mass stars undergo fusion at central temperatures close to 2.5×10^7 K. At this temperature, calculate the proton capture lifetimes of ^{12}C, ^{13}C, ^{14}N, and ^{15}N. Express your answer in years with ρ normalized to 1 g cm^{-3}. Which reaction in the CNO cycle is the rate limiting step under the conditions appropriate for stellar interiors?

   c) At high temperatures, the proton capture lifetimes decrease. At some temperature, the proton capture lifetime becomes shorter than the typical beta decay timescale of $\sim$ 100 s. At this point, beta decay, not proton capture, becomes the rate limiting step in the CNO cycle. For $\rho = 100$ and 10^5 g cm^{-3}, calculate the temperature at which this transition occurs. Around this temperature, fusion occurs via what is called the hot CNO cycle, which is very important in a number of astrophysical environments, e.g., nuclear fusion of accreted material on neutron stars (which is why the high density of 10^5 g cm^{-3} used above is relevant). Note that the sequence of reactions in the hot CNO cycle differs a bit from the CNO cycle in main sequence stars, but we won't worry about those difference here.

   d) In the limit that β decays are the rate limiting step in the CNO cycle (i.e., at high T), estimate the energy generation rate ϵ (ergs/s/g) in terms of the density ρ, temperature T, hydrogen mass fraction X and CNO mass fraction Z_{CNO}. Your answer may not depend on all of these quantities.

5. **Very Massive Stars:** Consider very massive stars with $M \sim 50 - 100 M_\odot$. Recall that I showed in lecture that in such stars, radiation pressure due to photons is more important than gas pressure. Fusion is by the CNO cycle. Assume for now that energy is transported primarily by photons and that the opacity is due to Thomson scattering.

   a) Estimate the fraction of the mass in the star that is undergoing convection (recall that fusion by the CNO cycle is very concentrated at small radii because of the strong temperature dependence). For comparison, detailed calculations show that the fraction of the mass that undergoes "core" convection increases from $\simeq 10\%$ at $2M_\odot$ to $\simeq 75\%$ at $60M_\odot$.

   b) Calculate the main sequence lifetime of a very massive star as a function of its mass M. Be sure to take into account the results of a).