Homework #4

1. **Free Electrons at Low Temperatures**

   a) Use the Saha equation to calculate the fraction of free electrons n_e/n_{tot} for a pure hydrogen gas. Plot your result as function of temperature. Assume $n_{tot} = 10^{17}$ cm^{-3} as is appropriate for the solar photosphere, where n_{tot} is the total number of nuclei (ionized + neutral).

   b) In a more realistic situation, both hydrogen and other elements contribute to the number of free electrons n_e that shows up in the Saha equation. Let's consider a simple model in which we have a gas of hydrogen and a single metal. The metal has an ionization energy $\chi_m \approx 5$ eV (as is appropriate for sodium). For simplicity, we will assume that the metal can only be ionized once (this is reasonable at the low temperatures where the presence of the metal is important). Assume that $n_{tot,m} \approx 10^{-6} n_{tot,H}$ (again reasonable for sodium) and set the product of g's for the metal to 1. Calculate the fraction of free electrons $n_e/n_{tot,H}$ as a function of temperature for the combined hydrogen + metal gas. As in a), assume $n_{tot,H} = 10^{17}$ cm^{-3}. Make a plot showing the ratio of n_e for part b) to n_e for part a) as a function of temperature. Below what temperature does the assumption of a pure hydrogen gas start to become inaccurate for predicting n_e? NOTE: Solving part b) will require a numerical root finding technique. You can use Fortran, C, IDL, Matlab, Mathematica, etc.

2. **Hydrogen Lines from Stars**

   a) Again assume a pure hydrogen gas. Use your result from 1a) to calculate the fraction of all H atoms that have an electron in the $n = 2$ state of hydrogen. If n_2 is the number density of atoms with electrons in the $n = 2$ state, then what we are after here is n_2/n_{tot}. You will need to use the Boltzmann factor in addition to your result from the Saha equation in 1a). Using the same density as in 1a), make a plot of n_2/n_{tot} as a function of stellar surface temperature. Recall that the energy levels of the H atom are given by $E = -13.6/n^2$ eV and the degeneracies are $g_n = 2n^2$.

   b) The Balmer lines of hydrogen are produced by transitions between the $n = 2$ states of Hydrogen and the $n = 3, 4, ...$ states. What are the wavelengths of the H_α ($n = 2 \rightarrow 3$) and H_β ($n = 2 \rightarrow 4$) lines of H? Using your result from a), which stellar spectral type should show the most prominent H lines? The right answer is A stars, with $T_{\rm eff} = 10000$ K. Although you will not be very far off, you should get the *wrong* answer in this part of the problem.

   c) The cross-section at line center for the production of Balmer lines is $\sigma \simeq 10^{-16}$ cm^2. This is the cross section for an atom with an electron in the $n = 2$ state to absorb a photon and be excited up to a higher energy state. Use this fact to explain why the answer in part b) is incorrect. Can you quantitatively explain why A stars with $T_{\rm eff} = 10000$ K show the most prominent H lines?

3. **Element Diffusion in Stars**

   At a number of points in lecture, I noted that mixing of elements by convection is important because it brings fresh material into regions where nuclear fusion occurs and because it can bring to the surface the products of nuclear burning that occurred deeper in the star. This naturally raises the question of whether other sources of element mixing are important in stars. In this problem we will assess this issue quantitatively for the case of gravitational settling, the tendency for heavier elements to sink in a gravitational field. Excluding part g, use reasonable solar values when making numerical estimates.

   a) Consider a star composed solely of ionized hydrogen. Charge neutrality requires that $n_e = n_p$ and thermal equilibrium implies $T_e = T_p$. Show that for both the electrons and protons to be in force balance (no net force), there must be an electric field with a magnitude $eE = (m_p - m_e)g/2 \approx m_p g/2$ where g is the local gravitational acceleration.

b) Consider a trace population of ions with number density n_{ion}, charge Ze and mass Am_p in our otherwise pure hydrogen star. What is the net macroscopic force per unit volume on the ions? What is its direction? Assume that the star is chemically homogeneous, i.e, that n_{ion}/n_H is independent of position.

c) In addition to the macroscopic force from b), the ions undergo Coulomb collisions with the protons and electrons in the star. Estimate how often an ion collides with electrons and protons. Which collisions – ion or electron – dominate the momentum exchange?

d) Combine your results from b) and c) to determine the drift velocity v_d between hydrogen and the ions as a function of the density, temperature, and gravity in the star.

e) For a typical ion (say fully ionized O), how does the time to drift a distance comparable to the radius of the star compare to the main sequence lifetime? Is settling important in the interiors of stars?

f) Let's assess the importance of diffusion in the photospheres of stars. How long does it take an ion to sink a distance comparable to the scale height of the photosphere of the sun? Should we see heavy elements in the photospheres of stars or just hydrogen?

g). Finally, carry out the same estimate as in f) for a $0.7M_\odot$ white dwarf with $T_{eff} \approx 20,000$ K and $R \approx 0.01R_\odot$. Should we see heavy elements in its photosphere or just hydrogen?