Homework #5

1. **The Helium Main Sequence**

   In certain (later) stages of stellar evolution, stars are largely composed of He and He fusion dominates the stellar luminosity. One can approximate such stars as lying on a He main sequence. In this problem we will calculate the properties of the He main sequence assuming that a star is composed of pure He, that energy transport is via radiation, that electron scattering dominates the opacity, and that gas pressure dominates. Note that throughout this problem you should not just give scaling laws for the desired relations; you should also determine reasonable normalizations.

   a. Calculate the mass-luminosity relationship for the He main sequence.

   b. Estimate the core temperature of a 1 solar mass He star. You do not need to do the full integral $L_{fusion} = \int dM\epsilon$, but can approximate this as $L_{fusion} \sim 0.1M\epsilon(r = 0)$. Given your result for T_c for a 1 $M_\odot$ star, calculate the power-law relation $T_c(M)$.

   c. Use your results above to determine the $R(M)$ and $T_{eff}(L)$ relations for the He main sequence. Then sketch the relative positions of the H & He main sequences in the HR diagram.

   d. What is the He main sequence lifetime as a function of stellar mass? Compare this to the corresponding H burning lifetime.

2. **The Mass-Radius Relation for White Dwarfs**[1]

   Equation 3.53 of HKT (3.50 of HK) gives the general expression for the pressure of a cold degenerate gas. In the extreme non-relativistic or relativistic limits, we can write the degenerate equation of state in polytropic form, but this is not possible in general.

   Write a program to solve the equations of hydrostatic equilibrium and $dM_r/dr = 4\pi r^2\rho$ using the general equation of state of a cold degenerate electron gas. This solution provides the density and pressure profiles and the mass-radius relationship for such objects. Boundary conditions on these differential equations only need to be provided at the center of the star so you can integrate from the center to the surface just like you did for polytropes in HW 1. You might find it easiest to write $dP/dr = (dP/d\rho)(d\rho/dr)$ and solve for $dP/d\rho$ from the EOS. Since $dP/d\rho$ is just a function of ρ (as you can show from the EOS), this technique eliminates P as an explicit variable that you have to worry about.

   a) By varying the central density you get a one-parameter sequence of stars with different masses M and radii R. Plot the resulting mass-radius relationship for $\mu_e = 2$ (a C/O WD) and $\mu_e = 56/26 = 2.15$ (an iron WD). Explain the behavior of your numerical results as $R \to 0$ and $R \to \infty$ in terms of the polytropic models discussed in class. In addition, for a $0.6M_\odot$ and $1M_\odot$ $\mu_e = 2$ WD, compare your numerical result to the $\gamma = 5/3$ polytropic result for R(M) derived in class.

   b) The results you have derived in a) should show that as $M \to 0$, $R \to \infty$. As we discussed in our lecture on brown dwarfs, however, this is not correct because Coulomb interactions become important in the equation of state of low-mass objects (brown dwarfs and planets). Estimate the density at which the Coulomb energy per particle becomes comparable to the Fermi energy. What mass and radius does this correspond to? This is a very rough estimate of the maximum radius of a degenerate object.

   c. Type 1a supernovae are believed to occur when carbon fusion begins in a WD under highly degenerate conditions and leads to the explosion of the star. This can occur if the mass of the WD approaches the Chandrasekhar mass (probably because of accretion, though models in which mergers of two WDs do the job have also been presented). As your calculations in a)

[1] Chandrasekhar Nobel Prize (1983)

show, as $M \to M_{ch}$, the central density of the WD increases. The central temperature does as well because of compressional heating. At masses sufficiently close to M_{ch}, carbon fusion begins when energy generation by fusion exceeds energy losses by neutrinos. The star explodes soon thereafter. Detailed models predict that carbon fusion begins when $\rho_c \approx 3 \times 10^9$ g cm^{-3} (and $T_c \approx 7 \times 10^8$ K, but the temperature is not important for this problem).

i) Use your results from a) to determine the mass and radius of the WD when it explodes. Assume $\mu_e = 2$.

ii) What is the net binding energy E_{tot} of the WD at this time (don't worry about the thermal energy, which is negligible)? Recall that when $\gamma \to 4/3$ $E_{tot} \to 0$. To quantify just how close the WD is to this limit, it is useful to compare the WD's binding energy to its gravitational potential energy.

iii) If the entire star undergoes carbon fusion, show that sufficient energy is released to unbind the entire star. Estimate the asymptotic kinetic energy and velocity of the ejecta.

3. **Zero Metallicity Stars**

Consider a primordial massive star ($M \sim 100 M_\odot$) formed at high redshift. It initially has no metals but has both hydrogen ($X \approx 0.75$) and helium ($Y \approx 0.25$). Recall that massive stars such as this one are supported by radiation pressure.

a) Show that He fusion, not H fusion via the pp chain, dominates the energy generation in such stars. Hint: Calculate the temperature at which each fusion process would produce the luminosity of the star, making reasonable assumptions about the radius/density.

b) Now estimate the L(M), $T_c(M)$, R(M), and $T_{eff}(L)$ relations for zero metallicity massive stars. As in problem 1, determine appropriate normalizations for all of your scaling relations.

c) Carbon and Oxygen are products of He fusion. Thus once He fusion has occurred for some amount of time, there will be enough C & O around for H fusion to proceed by the CNO cycle.

i) Estimate the CNO mass fraction Z_{CNO} required for CNO fusion to dominate over He fusion in the centers of primordial stars.

ii) Roughly how long does it take the star to generate this amount of CNO?

Rather counter-intuitively, what this means is that primordial massive stars spend the vast majority of their lives fusing $H \to He$ via the CNO cycle, using the trace amount of CNO generated by a brief epoch of He fusion!

iii) When fusion proceeds via the CNO cycle instead of He fusion, will your results for part b) change significantly? Why or why not? You don't need to give a detailed quantitative answer for this part of the problem, but just explain your answer physically.