Astronomy 7A: Introduction to Astrophysics

Final Exam SOLUTIONS

Your Name:

Your Section Date & Time:

Show your work and explain your answers (we will give partial credit but to do so we need to see your work). Do the problems in any order you wish. If you have any questions about the exam, ask us. You have 3 hours. **Read the questions carefully.**

1. **Spectra (15)**

   Above are two plots. The left panel shows the observed flux from an unresolved object as a function of frequency near the frequency ν_0 of an atomic transition. The right panel shows the cross section for absorption as a function of frequency near the same frequency ν_0. Explain carefully under what conditions an *optically thick* object would produce the flux shown and draw a plot of how the temperature in the atmosphere of the object changes with radius.

   The photosphere of the object must be hot at the base, then cool off as radius increases. The outer portion of the photosphere however must increase in temperature with radius so that the cool central photosphere in front of the hot base of the photosphere causes a large absorption feature, while the hot outer photosphere in front of the cooler central photosphere causes an emission feature on top of that.

2. **Cold Fusion! (18)**

   The gas at the center of a star must be sufficiently hot in order for fusion to occur. This is because particles need to have high kinetic energy in order to effectively tunnel through the Coulomb barrier. Recall from class that the approximate quantitative condition for fusion to occur is that the de Broglie wavelength of ions λ must be larger than the classical distance of closest approach r_c of two ions interacting via the Coulomb force.

   Now consider a low temperature degenerate gas of fermions. Because fermions have kinetic energy even at zero temperature it is still possible for the ions to tunnel through the Coulomb barrier in a very cold, but also very degenerate, gas. This leads to cold fusion (the technical name is pychnonuclear reactions)!

   a) Consider two ions, each with Z protons and mass number A. The ions have a total energy E. The ions are interacting solely via the electric force. Calculate the closest the two ions can get to each other, the classical distance of closest approach r_c, because of the repulsive Coulomb interaction. (4)

   We set the total energy E equal to the electric potential energy at r_c, the distance of closest approach:

   $$E = \frac{kZ^2e^2}{r_c}$$

   $$r_c = \frac{kZ^2e^2}{E}$$

   b) Use the expression for the energy of a particle E in a degenerate gas to rewrite your expression in a) for the classical distance of closest approach r_c in terms of the ion density n_i. The ions are non-relativistic. (3)

Plugging in the expression for E_F from the equation sheet, with $m = Am_p$,

$$r_c = \frac{2Am_p k Z^2 e^2}{\hbar^2 (3\pi^2 n_i)^{2/3}}$$

c) Show that an approximate condition for significant tunneling through the Coulomb barrier – and thus for fusion – in a cold degenerate gas is that the ion number density n_i exceed a critical value. You should derive an expression for the critical density n_i in terms of fundamental constants and the charge Z and mass number A of the ions. (8)

We get significant tunneling if $\lambda \geq r_c$. $\lambda \sim h/p$, so

$$\begin{aligned}
\frac{h}{\hbar n_i^{1/3}} &\geq \frac{2Am_p k Z^2 e^2}{\hbar^2 (3\pi^2 n_i)^{2/3}} \\
\frac{2\pi\hbar}{\hbar} n_i^{1/3} &\geq \frac{2Am_p k Z^2 e^2}{\hbar^2 (3\pi^2)^{2/3}} \\
n_i^{1/3} &\geq \frac{Am_p k Z^2 e^2}{\pi\hbar^2 (3\pi^2)^{2/3}} \\
n_i &\geq \left(\frac{Am_p k Z^2 e^2}{\pi\hbar^2}\right)^3 \frac{1}{(3\pi^2)^2}
\end{aligned}$$

d) For Carbon ions (Z = 6; A = 12), what is the critical density n_i (in m^{-3})? In what type of star might the conditions for cold fusion be realized? (3)

For $Z = 6$, $A = 12$, the critical density from (c) is $n_i \approx 10^{44}\ m^{-3}$. Compare this to average densities for a carbon white dwarf and a neutron star:

$$n_{WD} \approx \frac{3M_\odot}{4\pi(10^7\ m)^3 12m_p} \approx 2 \times 10^{34}\ m^{-3}$$

$$n_{NS} \approx \frac{3 \times 1.4M_\odot}{4\pi(10^4\ m)^3 m_n} \approx 4 \times 10^{44}\ m^{-3}$$

So these conditions for cold fusion might be realized in a neutron star! (More detailed calculations actually show that cold fusion could be important in WDs.

3. **Detecting Neutrinos (17)**

A supernova is an explosion at the end of a massive star's life when fusion can no longer supply energy to support the star against gravity. The energy released in the explosion is E_{SN}, nearly all of which comes out in neutrinos that have a typical energy E_ν.

a) How many neutrinos are produced in the explosion? (2)

$$N \sim \frac{E_{SN}}{E_\nu}$$

Consider a detector deep under the surface of the Earth that is trying to find neutrinos from a supernova a distance D away. The detector is a cube with sides of length ℓ that is filled with water of mass density ρ. Neutrinos are detected when they scatter, with cross-section σ, off of a water molecule. Recall that the mass of a water molecule is $m_w = 18\ m_{proton}$.

b) How many neutrinos pass through the detector during the explosion without scattering off of a water molecule? (5)

The number of neutrinos that go to the detector is $\frac{Nl^2}{4\pi D^2}$. The optical depth of the detector τ to the neutrinos is $\tau = \rho\sigma l/m_w$. So the number of neutrinos that pass through the detector is

$$N_{pass} = \frac{Nl^2}{4\pi D^2} e^{-\rho\sigma l/m_w}$$

c) How many neutrinos are actually detected during the explosion? (6)

$$N_{detected} = \frac{Nl^2}{4\pi D^2}\left(1 - e^{-\rho\sigma l/m_w}\right)$$

d) Give numbers for b) and c) if $E_{SN} \approx 10^{46}$ J, $E_\nu \approx 10$ Mev, $\ell \approx 30$ m, $\sigma \approx 10^{-46}$ m^2, and $D \approx 100$ kpc $\approx 3 \times 10^{21}$ m. In case you forgot, the density of water is $\rho = 10^3$ kg m^{-3}. (4)

With these numbers, $N \sim 6 \times 10^{57}$, $\tau \sim 10^{-16}$, so $N_{pass} \sim 5 \times 10^{16}$ and $N_{detected} \sim 5$.

4. **Concepts (17)**

a) Will most stars end up as WDs, NSs, or BHs? Explain your answer. (4)

Most will end up as white dwarfs since low mass stars end up as white dwarfs and most stars are low mass.

b) Do white dwarfs and neutron stars have a mass-luminosity relationship the way that main sequence stars do? Why or why not? (4)

No - white dwarfs and neutron stars don't have a mass-luminosity relationship since they just cool off as they age, so that their luminosity decreases but their mass stays the same.

c) Explain why the Balmer lines of H become progressively less prominent in the spectra of main sequence stars with $M < M_\odot$. Recall that the Balmer lines are produced by transitions between the $n = 2$ and higher states of H. (4)

Balmer lines become progressively less prominent because as you look at lower and lower mass stars, T_{eff} gets lower so that (according to the Boltzmann equation) an increasingly large fraction of Hydrogen is in the ground state and not the higher energy levels, so weaker Balmer lines, which depend on Hydrogen in the n=2 state.

d) Consider a Carbon and Oxygen white dwarf with $M = M_\odot$ and $R = 10^7$ m. The efficiency of Carbon and Oxygen fusion is $\simeq 0.07\%$, 10 times smaller than that of hydrogen fusion. If the entire white dwarf were to be fused to heavier elements, would the star remain bound or would it explode like a bomb? (5)

The gravitational potential binding energy is roughly $U \sim GM^2/R \sim 3 \times 10^{43}$ J while the energy from fusion is $E_{fusion} \sim 7 \times 10^{-4} Mc^2 \sim 10^{44}$ J. So the energy from fusion is greater than the binding energy so the white dwarf would explode.

5. **Hydrostatic Equilibrium (17)**

Consider an object of radius R and mass M supported by a hypothetical pressure given by $P = K\rho^\gamma$ where K and γ are constants.

a) Use hydrostatic equilibrium to derive an order of magnitude estimate of the radius R as a function of mass M. Note that you should give a full expression for the radius, including constants, not just a scaling relationship. (9)

$$\begin{aligned}
\frac{dP}{dr} &= -\rho\frac{GM_r}{r^2} \\
\frac{P}{R} &\approx \rho\frac{GM}{R^2} \\
K\rho^{\gamma} &\approx \rho\frac{GM}{R} \\
K\rho^{\gamma-1} &\approx \frac{GM}{R} \\
K\left(\frac{3M}{4\pi R^3}\right)^{\gamma-1} &\approx \frac{GM}{R} \\
R &\approx M^{\frac{\gamma-2}{3\gamma-4}}\left(\frac{K}{G}\right)^{\frac{1}{3\gamma-4}}\left(\frac{3}{4\pi}\right)^{\frac{\gamma-1}{3\gamma-4}}
\end{aligned}$$

b) Show that for γ less than a critical value, i.e., for $\gamma < \gamma_{crit}$, hydrostatic equilibrium is *unstable*. You should calculate the value of γ_{crit} and explain in a few sentences why hydrostatic equilibrium is unstable for $\gamma < \gamma_{crit}$. HINT: Determine how the pressure force and gravitational force depend on the mass M and radius R of the object. Now consider how these forces would change if there were a small decrease in the radius of the star. (8)

$$\begin{aligned}
\frac{K\rho^{\gamma}}{R} &\approx \rho\frac{GM}{R^2} \\
K\left(\frac{3}{4\pi}\right)^{\gamma}\frac{M^{\gamma}}{R^{3\gamma+1}} &\approx \frac{3GM^2}{4\pi R^5}
\end{aligned}$$

If M is constant but R decreases by a small amount, this equilibrium will be stable if the pressure force increases a little more than the gravitational force increases so that the slightly larger pressure force will push the star back out to its equilibrium radius. Conversely, the equilibrium will be unstable if the gravitational force increases slightly more than the pressure force does. This unstable case will happen if the exponent on R in the denominator is larger for the gravitational force than the pressure force - i.e.

$$5 > 3\gamma + 1$$

$$\gamma < \frac{4}{3} = \gamma_{crit}$$

6. **Accreting White Dwarfs (16)**

Consider a WD of mass $M \approx 1\,M_{\odot}$ and radius $R \approx 10^7$ m accreting at a rate $\dot{M}$. For simplicity, assume that the accreted gas is composed solely of hydrogen.

a) Assume that the hydrogen that settles onto the WD is steadily (continuously) fused into helium. Under these circumstances, would the luminosity of the system be dominated by the energy produced by the accretion disk around the WD or the energy produced by fusion on the surface of the WD? (3)

$$L_{fusion} \sim 0.007 \times \dot{M}c^2 \sim 6 \times 10^{14} \left(\frac{\dot{M}}{kg\ s^{-1}} \right)\ J\ s^{-1}$$

$$L_{disk} \sim \frac{GM\dot{M}}{2R} \sim 7 \times 10^{12} \left(\frac{\dot{M}}{kg\ s^{-1}} \right)\ J\ s^{-1}$$

So the luminosity would be dominated by the fusion on the surface of the white dwarf (~ 100 times more luminous than the accretion disk).

At accretion rates $\dot{M} \approx 10^{-8} M_\odot$ yr^{-1}, the hydrogen that settles onto the WD cannot be steadily fused into helium. Instead the material builds up on the white dwarf surface. When the temperature of the accreted hydrogen layer becomes high enough fusion sets in under runaway conditions (like a bomb) and the accumulated hydrogen is rapidly fused. This rapid fusion produces a bright flash of radiation called a classical nova. The nova produces a luminosity $L \approx 10^{31}$ W and lasts for $\Delta t \approx 100$ days.

b) Compare the luminosity of the nova to the luminosity produced by the accretion disk around the white dwarf. Which is larger and by how much? (3)

Plugging $\dot{M} \approx 10^{-8}\ M_\odot\ yr^{-1}$ into our expression in (a) for the luminosity of the disk, we find $L_{disk} \approx 4.2 \times 10^{27}\ J\ s^{-1}$. Thus the nova is about 2400 times brighter than the luminosity produced by the accretion disk.

c) In what part of the electromagnetic spectrum would the nova produce most of its radiation? (3)

For blackbody radiation,

$$L = 4\pi R^2 \sigma T^4 = 10^{31}\ W$$

$$T = \left(\frac{10^{31}\ W}{4\pi (10^7\ m)^2 \sigma} \right)^{1/4} \approx 6 \times 10^5\ K$$

Plugging this temperature into Wien's law,

$$\lambda_{max} = \frac{0.0029\ m\ K}{6 \times 10^5\ K} \approx 5\ nm$$

This is in the xray regime, but UV is also an acceptable answer.

d) Classical nova are recurrent phenomena (i.e., they occur over and over again). Use the information given before part b) to estimate the typical time between nova on the surface of the WD. Assume that 1/2 of the energy produced by fusion comes out as radiation and that only 10% of the accumulated hydrogen undergoes fusion during a nova (most is actually blown away). (4)

If t is the time between novas during which the mass for the nova builds up, from the information given we can write

$$E_{nova} = 0.5 \times 0.007 \times 0.1(\dot{M}t)c^2$$

Also, the energy released by the nova is

$$E_{nova} = (10^{31}\ W) \times (100\ days) \approx 8.6 \times 10^{37}\ J$$

Setting these expressions equal, we find

$$t = \frac{8.6 \times 10^{37}\ J}{0.5 \times 0.007 \times 0.1\dot{M}c^2} \approx 140\ years$$

e) An amount of mass m_{fuse} is fused from H into He during the nova. Assume that 1/2 of the energy released by fusion during the nova goes into just unbinding matter off the surface of the WD. How much mass is blown off the surface of the WD during the nova (your answer should be in terms of m_{fuse})? (3)

The energy going into unbinding matter from the surface is $\frac{1}{2} \times 0.007 m_{fuse} c^2$. The energy it would take to unbind mass m_{eject} from the surface is GMm_{eject}/R. Setting these equal, we find

$$m_{eject} = \frac{0.007 m_{fuse} c^2 R}{2GM} \approx 24 m_{fuse}$$