RADIATIVE TRANSFER

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1. INTRODUCTION AND BASIC DEFINITIONS

This discussion is based on a combination of Hollenbach and McKee (1979 ApJSuppl 41, 573; HM); R. Genzel (1991, in The Talactic Interstellar Mediujm Saas Fe Lectures, ed. Burton, Genzel, Elmegreen; RG); Scoville and Solomon (1974 ApJ 187, L67); Mihalis (1978: Stellar Atmospheres); Ferland & Osterbrock (2006 textbook); the RADEX manual (van Langevelde & van der Tak 2008).

Until now, we have considered optically thin cases in which the brightness (specific intensity I) increases linearly with the product of path length times volume emissivity. Clearly, though, this can't go on forever or else we'd get infinitely high brightnesses, or if not infinitely high, exceeding the blackbody radiation field. So we must consider the effects of opacity, usually called by the misnomer "optical depth" (opacity effects occur at all wavelengths, not just optical!).

We write the equation of transfer

$$\frac{dI_\nu}{ds} = \epsilon_\nu - \kappa_\nu I_\nu \tag{1}$$

where ds is positive *towards the observer*, and we normally define two quantities, the *optical depth* (opacity) τ_ν and the *source function* Σ_ν:

$$d\tau_\nu = -\kappa ds \tag{2a}$$

$$\Sigma_\nu = \frac{\epsilon_\nu}{\kappa_\nu} \tag{2b}$$

Notice that $d\tau$ is positive *away from the observer.* That is, we speak of the front surface of a cloud, or star, as as having optical depth zero, while somewhere in the deep interior of a cloud has $s = 0$. With this, the equation of transfer becomes

$$\frac{dI_\nu}{-d\tau_\nu} = \Sigma_\nu - I_\nu \tag{3}$$

For a discussion of general solutions, see Mihalis.

Consider the LTE case in which the emission process is described by a single temperature T. Then $\Sigma_\nu = B_\nu(T)$. In realistic ISM conditions, this temperature is not necessarily the kinetic temperature because collisions may not dominate the distribution. So, more generally, we define the *excitation temperature* T_x as *that temperature that gives us the proper population ratio* $\frac{n_2}{n_1}$. (Here we consider a two-level system with the upper level being 2 and the lower 1). Thus,

$$\Sigma_\nu = B_\nu(T_x) \tag{4}$$

In the case of a single two-level system, we do *not* need $T_x = T_K$; in the case of a multiple level system, such as a molecule, *each pair of levels can have a different* T_x *and, moreover, the ratios* $\frac{n_3}{n_1}$ *and* $\frac{n_3}{n_2}$ *can have different* T_x*!* So this use of T_x is completely general.

2. EXPRESSING IN TERMS OF EINSTEIN COEFFICIENTS

2.1. Some Important Relationships among Einstein Coefficients

The standard relationships among the Einstein coefficients are

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21} \tag{5a}$$

$$\frac{B_{21}}{g_1} = \frac{B_{12}}{g_2} \tag{5b}$$

Now define the energy of the emitted photon in temperature units, the *transition temperature* is

$$T_{21} = \frac{h\nu}{k} \tag{6}$$

and consider an atom sitting in a blackbody radiation field whose temperature is T_{21}. The ratio of the downward radiatively induced rate to the downward spontaneous rate is

$$\frac{B_{21}J}{A_{21}} = \frac{B_{21}B_\nu(T_{21})}{A_{21}} = \frac{1}{e^1 - 1} = 0.6 \tag{7}$$

Thus we reach the important conclusion that *for a transition in a a radiation field having* $J = B_\nu(T_{21})$, *the induced rate is nearly equal to the spontaneous rate.* Of course, this conclusion is hardly new: it appears in the basic reasoning Einstein used to derive his famous coefficients.

2.2. Emission and Absorption coefficients in terms of Einstein Coefficients

We are interested in spectral lines, not continuum. So we can express ϵ and κ in terms of the Einstein coefficients. For a spectral line, κ_ν contains the information on line shape. Einstein coefficients give total emission/absorption integrated over the whole line, so they tell us $\kappa_\nu d\nu$. Then

$$\int \epsilon_\nu d\nu = \frac{h\nu}{4\pi} n_2 A_{21} \tag{8a}$$

expresses the rate of photon emission per steradian times the photon energy, and

$$\int \kappa_\nu d\nu = \frac{h\nu}{4\pi}(n_1 B_{12} - n_2 B_{21}) \tag{8b}$$

expresses absorptions minus stimulated emissions: κ is the *net* absorption, *accounting for stimulated emission from the upper level.* This means that κ depends on T_x.

With the standard relationships among the Einstein coefficients, and the Boltzmann distribution $\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu/kT_x}$ (Note the x in T_x !!), we have

$$\int \kappa_\nu d\nu = n_2 \frac{A_{21} c^2}{8\pi\nu^2}(e^{h\nu/kT_x} - 1) \tag{9a}$$

$$\int \kappa_\nu d\nu = n_{tot} \frac{A_{21} c^2}{8\pi\nu^2} \frac{g_2}{g_1} \frac{1 - e^{-h\nu/kT_x}}{1 + \frac{g_2}{g_1} e^{-h\nu/kT_x}} \tag{9b}$$

There is some important behavior to notice in the above equation. If $T_x \to \infty$, then $\kappa \to 0$; in this limit, stimulated emissions just cancel absorptions so $\kappa \to 0$. If $T_x \to 0$, then all the atoms go to the ground state 1, so there are no stimulated emissions and

$$\int \kappa_{\nu, T_x=0} d\nu = n_{tot} \frac{A_{21} c^2}{8\pi\nu^2} \frac{g_2}{g_1} \tag{10}$$

Finally, *negative temperatures* aren't excluded: they correspond to $\frac{n_2}{n_1} > \frac{g_2}{g_1}$—the case of interstellar *masers*, with $\kappa < 0$.

Now write the source function Σ: you find that n_2, A_{21}, and ϕ_ν all cancel out so that

$$\Sigma_\nu = B_\nu(T_x) \tag{11}$$

This simply reflects the fact that, by defining $\frac{n_2}{n_1}$ in terms of a temperature, we are in effect assuming LTE; and in LTE the source function is always $\Sigma_\nu = B_\nu(T_x)$. In particular, there are no line parameters (Einstein A, shape function) in Σ!

3. EXPRESSING TEMPERATURES IN TERMS OF THE EQUIVALENT RAYLEIGH-JEANS TEMPERATURE T_{RJ}

We always encounter the lengthy expressions of the sort $\left[\frac{2h\nu^3}{c^2}(e^{h\nu/kT}-1)^{-1}\right]$, which makes equations cumbersome. Here, T might be a brightness temperature T_B, an excitation temperature T_x, or a kinetic temperature T_K. In the RJ limit this simplifies to $\left[\frac{2kT\nu^2}{c^2}\right]$. This makes it convenient to follow Genzel and define the *equivalent Rayleigh-Jeans temperature* T_{RJ}. With this,

$$\frac{kT_{RJ}}{h\nu} = \frac{1}{e^{h\nu/kT}-1} \tag{12a}$$

$$\frac{T_{RJ}}{T_{21}} = \frac{1}{e^{T_{21}/T}-1} \tag{12b}$$

$$B_\nu(T) = \frac{2k\nu^2 T_{RJ}}{c^2} = \frac{2kT_{RJ}}{\lambda^2} \tag{12c}$$

In the RJ limit $\frac{T_{21}}{T} \lesssim 1$, $T_{RJ} \rightarrow T$. In the Wein limit $\frac{T_{21}}{T} \gtrsim 1$, the exponential dominates, and $T_{RJ} \rightarrow T_{21}e^{-T_{21}/T}$.

Figure 1 shows this somewhat complicated function for the important case of the CO (1-0) line, for which $T_{21} = 5.53$ K. Looking at this, one sees huge nonlinearity for $T_B \lesssim 2.5$ K, so weak lines would seem to be grossly affected. However, not to worry! The minimum possible T_B is the CBR, 2.73 K. All line intensities add on to this continuum brightness. When you observe a line, you are observing the *difference between the (line plus continuum) and the (continuum).* So an apparently weak line of apparent brightness of, say, 0.1 K actually has brightnes 2.83 K; you notice the difference. Note, also, the constant difference as the RJ limit is approached (i.e., large T_B). The difference is $T_{21}/2$ and results from the second-order term in the Taylor expansion for $\exp(T_{21}/T_B)$.

With this simple way of writing things, we have for the absorption coefficient:

$$\int \kappa_\nu d\nu = n_2 \frac{A_{21}c^2}{8\pi\nu^2} \frac{T_{21}}{T_{RJ,x}} \tag{13a}$$

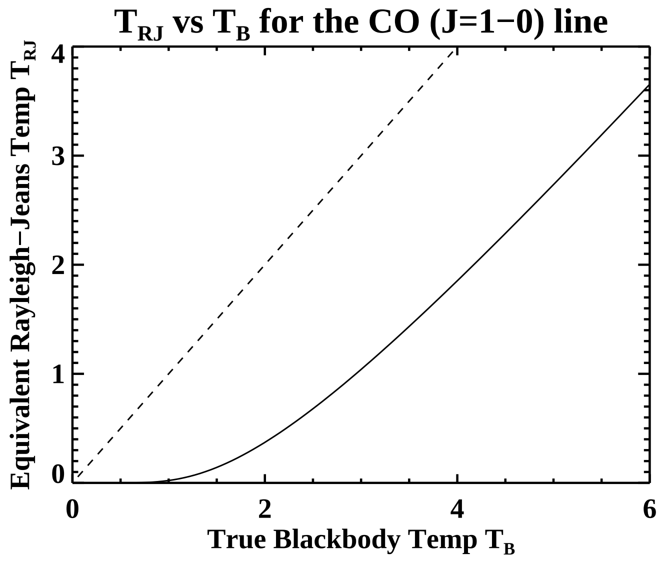

Fig. 1.— The Rayleigh-Jeans equibalent temperature T_{RJ} versus the actual true blackbody temperature T_B for the [CO (J=1-0)] line, for which $T_{21} = 5.53$ K. The dashed line has $T_{RJ} = T_B$.

$$\int \kappa_\nu d\nu = n_{tot} \frac{A_{21} c^2}{8\pi\nu^2} \frac{T_{21}}{T_{RJ,x}} \frac{\frac{g_2}{g_1}}{1 + \frac{g_2}{g_1} + \frac{T_{21}}{T_{RJ,x}}} \tag{13b}$$

The last factor is just $\frac{n_2}{n_{tot}}$ and is a bit cumbersome. Nevertheless, it is worth looking at this equation in the RJ limit, which corresponds to the important case of the 21-cm line (and many other radio lines, for that matter). Here, the last factor$\rightarrow \frac{1}{(g_1/g_2+1)}$ and $\kappa \propto \frac{1}{T_x}$, so that *cold clouds have higher optical depths*. This happens simply because the upper state gets less populated at colder temperatures, reducing the ratio of stimulated emissions to absorptions.

For the source function and specific intensity, we have:

$$\Sigma_\nu = \frac{2kT_{RJ,x}}{\lambda^2} \tag{14a}$$

$$I_\nu = \frac{2kT_{RJ,B}}{\lambda^2} \tag{14b}$$

so we can write for equation 3, the fundamental equation of transfer,

$$\frac{dT_{RJ,B}}{-d\tau_\nu} = T_{RJ,x} - T_{RJ,B} \tag{15}$$

4. THE LINE SHAPE FUNCTION ϕ_ν

Let ϕ_ν be the *line shape function.* It is the probability per unit frequency interval that the photon is emitted; $\int \phi_\nu d\nu = 1$. As ϕ_ν gets narrower, the line-center opacity increases: $\phi_{\nu,LC} \approx \frac{1}{\delta\nu}$, where $\delta\nu$ is the line width. This means $\kappa_\nu \sim \kappa_{LC}\delta\nu\phi_\nu$, where κ_{LC} is the opacity at line center.

Lines are commonly represented by Gaussians; if thermal broadening alone determines line shape, this is exact. Sometimes it is also important to include the Lorentzian "damping wings" or pressure broadening; the combination of a Gaussian and a Lorentzian is a Voigt profile (see RL). We, however, will stick with Gaussians. For a Gaussian,

$$\phi_\nu = \frac{1}{\sqrt{\pi}\delta\nu} e^{\frac{-\Delta\nu^2}{\delta\nu^2}} \tag{16a}$$

and

$$\tau_\nu = \tau_{LC} e^{\frac{-\Delta\nu^2}{\delta\nu^2}} = \sqrt{\pi}\tau_{LC}\delta\nu\phi_\nu \tag{16b}$$

where τ_{LC} is the optical depth at line center, $\Delta\nu$ is the frequency offset from line center and $\delta\nu$ is half the full $\frac{1}{e}$ width. Observers usually use the full width at half maximum $\delta\nu_{FWHM}$, for which

$$\delta\nu_{FWHM} = 2(\ln 2)^{1/2}\delta\nu = 1.665\delta\nu \tag{17a}$$

$$\int \tau d\nu = \frac{\sqrt{\pi}}{2(\ln 2)^{1/2}} \tau_{LC}\delta\nu_{FWHM} = 1.065\tau_{LC}\delta\nu_{FWHM} \tag{17b}$$

In terms of velocity V (km s^{-1}; $\delta V = \lambda\delta\nu$),

$$\delta V_{FWHM} = 0.213\sqrt{\frac{T}{A}} \text{ km s}^{-1} \tag{18a}$$

$$T = 21.8\ \delta V_{FWHM}^2 A \tag{18b}$$

5. THE INTEGRATED OPTICAL DEPTH

Define $N_{T_x=0}$ to be the total column density *assuming all species are in the lower (1) state* (which corresponds to $T_x = 0$). Then, rewriting equation 10,

$$\int \tau_{\nu,T_x=0} d\nu = N_{T_x=0} \frac{A_{21}c^2}{8\pi\nu^2} \frac{g_2}{g_1} \tag{19}$$

We usually express column densities in terms of the H-nuclei column density N(H); then $N_{tot} = \mathcal{A}N(H)$, where where $\mathcal{A}$ is the cosmic abundance of the element *in the gaseous phase* (some atoms are depleted onto dust). We define the *reference H-column density* $N(H)_{ref}$ as that H-nuclei column density for which the line-center optical depth is unity, i.e. $\tau_{LC,T_x=0} = 1$, for the particular case $\delta V_{FWHM} = 1$ km sec^{-1}.Then we have, from the above,

$$N(H)_{ref} = \frac{2.68 \times 10^{18}}{\mathcal{A}} \frac{g_1}{g_2} \frac{1}{A_{21}\lambda_\mu^3} \text{ cm}^{-2} \tag{20}$$

The line center optical depth becomes, in terms of T_x,

$$\tau_{LC} = \frac{N(H)}{N(H)_{ref}\delta V_{FWHM}} \frac{1 - e^{-T_{21}/T_x}}{1 + \frac{g_2}{g_1} e^{-T_{21}/T_x}} \tag{21a}$$

$$\tau_{LC} = \frac{N(H)}{N(H)_{ref}\delta V_{FWHM}} \frac{\frac{T_{21}}{T_{RJ,x}}}{\frac{T_{21}}{T_{RJ,x}} + \frac{g_2}{g_1} + 1} \tag{21b}$$

5.1. Some important lines and species

First, the 21-cm line, for which $A_{21} = 2.85 \times 10^{-15}$ s^{-1}, $\frac{g_2}{g_1} = \frac{3}{1}$, and $T_{21} = 0.068$ K.:

$$N(H)_{ref,HI\ 21cm} = 3.34 \times 10^{16} \text{ cm}^{-2} \tag{22}$$

For the two most important FS lines in PDR's:

$$N(H)_{ref,CII\ 157\mu m} = 3.56 \times 10^{20}\ \delta^{-1} \text{ cm}^{-2} \tag{23a}$$

$$N(H)_{ref,OI\ 63\mu m} = 3.14 \times 10^{20}\ \delta^{-1} \text{ cm}^{-2} \tag{23b}$$

where we take the atomic parameters from Genzel. These lines have $\frac{g_1}{g_2} = \frac{2}{4}$ and $\frac{5}{3}$. We take $\mathcal{A} = (6.3 \times 10^{-4}, 4.0 \times 10^{-4})$ for (O, C). These are the *undepleted* cosmic abundances from Spitzer. Using undepleted abundances is logically inconsistent in a PDR, where dust grains certainly exist and, indeed, have probably had time to grow by accreting even more interstellar gas atoms that in normal environments. Thus, the true $N(H)_{ref}$ are larger than the numerical values given above by the inverse of the depletion factor δ^{-1}, as written; the total gaseous abundance is $\mathcal{A}\delta$. The T_{21}'s for these lines are 92 and 230 K; the low T_{21} for CII makes it the most important coolant for the diffuse cold atomic ISM, where the temperature is too low to excite OI. The critical densities are ~ 2800 and 4.7×10^5 cm^{-3}; the high value for OI makes it the most powerful atomic coolant in the dense PDR's that occur next to HII regions. Of somewhat lesser importance is the related OI line,

$$N(H)_{ref,OI\ 145\mu m} = 2.46 \times 10^{20}\ \delta^{-1}\ \mathrm{cm}^{-2} \tag{23c}$$

Finally, for dust (which is continuum, not line), we have (for producing optical depth $\tau = 1$ at the specified wavelength):

$$N(H)_{ref,V} = 1.8 \times 10^{21}\ \mathrm{cm}^{-2} \tag{24a}$$

$$N(H)_{ref,1000A} = 4.2 \times 10^{20}\ \mathrm{cm}^{-2} \tag{24b}$$

Dust extinction in the FIR is negligible for our purposes.

5.2. Optical depths: HI versus CI in the CNM and WNM

Let's take a look at the optical depths of the 21-cm and 158 μm lines. Using equations 22 and 21a, we have

$$\tau_{LC,HI\ 21cm} = \frac{N(HI)}{1.96 \times 10^{18}} \frac{1}{\delta V_{FWHM} T_x} \tag{25a}$$

At a typical CNM temperature of 50 K,

$$\tau_{LC,HI\ 21cm} = \frac{N(HI)}{9.8 \times 10^{19}} \frac{1}{\delta V_{FWHM} (T_x/50)} \tag{25b}$$

For the 158 μm line, CNM volume density should be much smaller than the critical density, so $T_x \ll T_{21}$, in which case the ratio of exponentials on the right-hand side of equation 21a $\rightarrow 1$, so using equation 23a we have

$$\tau_{LC,CII\ 158\mu m} = \frac{N(HI)}{3.56 \times 10^{20}} \frac{\delta}{\delta V_{FWHM}} \tag{26}$$

This equation shows that the 158 μm line can easily be optically thick, because column densities $N(HI) > 3.56 \times 10^{20}\ \mathrm{cm}^{-2}$ are not unusual, particularly along lines of sight in the Galactic plane.

These two important lines have comparable optical depths. Specifically,

$$\frac{\tau_{LC,CII\ 158\mu m}}{\tau_{LC,HI\ 21cm}} = 0.27\delta \frac{T_x}{50} \tag{27}$$

In particular, for the warmer CNM clouds the optical depths become more comparable. The WNM is an interesting case: the 21-cm line optical depth is small because T_x is high, and per unit N(HI) the 158 μm optical depth is independent of T_x, so the CII line is much more optically think than the 21-cm line.

6. SOLUTION OF RADIATIVE TRANSFER FOR TWO SIMPLE CASES WITH KNOWN T_x

Suppose we know T_x as a function of z, or equivalently τ; then one can explicitly solve equation 15. As we shall see, this happens only under two circumstances: (1) collisions dominate ($\frac{n_{crit}}{n_{coll}} \ll 1$) and (2) collisions don't dominate but $\tau_{LC} \ll 1$. In both cases, as we shall see, T_x depends only on T_K and $\frac{n_{crit}}{n_{coll}}$.

In the nice case of a slab in which T_K, and therefore T_x, is constant, we have

$$T_{RJ,B} = T_{RJ,x}(1 - e^{-\tau_\nu}) + T_{RJ,B,BKGND}e^{-\tau_\nu} \tag{28a}$$

which has the nice simple interpretation: the first term is the emission within the slab; the second term is the emission incident from behind, attenuated by the opacity of the slab.

The line intensity is usually measured with respect to the surrounding continuum. If $T_{RJ,B,BKGND}$ is frequency-independent continuum, denoted by $T_{RJ,B,BC}$ (for **B**ackground **C**ontinuum), then the *apparent* line intensity is

$$T_{RJ,B,APP} = T_{RJ,B} - T_{RJ,B,BC} = (T_{RJ,x} - T_{RJ,B,BC})(1 - e^{-\tau_\nu}) \tag{28b}$$

Note that we have either an *emission* or *absorption* line, depending on the sign of $(T_{RJ,x} - T_{RJ,B,BC})$. In other words, cold clouds produce absorption lines.

6.1. Collisionally dominated: e.g. the 21-cm line

This simple case applies to the 21-cm line, for which $T_x = T_K$ because the critical density n_{crit} is very small. Moreover, because of the low frequency ($T_{21} = 0.068$ K) all the T_{RJ}'s become just plain T's and, in particular, $T_{RJ,B}$ becomes the standard brightness temperature T_B. Using this in equation 22, we have the interesting limits, first for the combination ($\tau_{LC} \ll 1$) and ($T_{B,BC} \ll T_K$):

$$T_{B,APP} = T_{B,LC} - T_{B,BC} \rightarrow T_K \tau_{LC} = \frac{N(HI)}{1.96 \times 10^{18} \delta V_{FWHM}} \tag{29a}$$

$$\int T_{B,APP} dV \rightarrow \frac{N(HI)}{1.83 \times 10^{18}} \tag{29b}$$

which means that the integrated line intensity $\propto$ the HI column density and is independent of T_K. This, plus the fortunate circumstances that the 21-cm line is, in fact, usually fairly optically thin and $T_{B,BC}$ is small, are of crucial importance for 21-cm line surveys: they provide the total HI column density.

The other interesting limit is, of course, $\tau_{LC} \gg 1$:

$$T_{B,LC} \rightarrow T_K \tag{29c}$$

so it's equivalent to being inside a blackbody at temperature T_K—as it must be. Note that $T_{B,LC} - T_{B,BC} = T_K - T_{B,BC}$: the line can be in absorption or emission, but in both cases case $T_{B,LC} \rightarrow T_K$, independent of $T_{B,BC}$—which, of course, makes sense because the background continuum is completely absorbed.

6.2. Collisions don't dominate but $\tau_{LC} \ll 1$

This case is characterized by the important necessity that *induced radiative transitions can be ignored.* That is, only two types of transition are important: collisions and spontaneous photon emission.

6.2.1. Let's calculate T_x

Define the upward collisional rate to be $n_1 n_{coll} \gamma_{12}$ and the lower to be $n_2 n_{coll} \gamma_{21}$ cm^{-3} sec^{-1}. Here n_{coll} is the volume density of colliding particles; for example, in an HII region they are usually electrons and in a neutral region hydrogen atoms or molecules. LTE considerations mean that $\frac{\gamma_{12}}{\gamma_{21}} = \frac{g_2}{g_1} e^{-h\nu/kT_K}$. In this case the equation of statistical equilibrium is

$$n_1 n_{coll} \gamma_{12} = n_2 (n_{coll} \gamma_{21} + A_{21}) \tag{30}$$

The solution of this is just

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-T_{21}/T_x} = \frac{g_2}{g_1} \frac{e^{-T_{21}/T_K}}{1 + \frac{n_{crit}}{n_{coll}}} = \frac{g_2}{g_1} \frac{1}{\left(1 + \frac{n_{crit}}{n_{coll}}\right)\left(1 + \frac{T_{21}}{T_{RJ,K}}\right)} \tag{31a}$$

$$T_{RJ,x} = \frac{T_{RJ,K}}{1 + \frac{n_{crit}}{n_{coll}}\left(1 + \frac{T_{RJ,K}}{T_{21}}\right)} \tag{31b}$$

where

$$n_{crit} = \frac{A_{21}}{\gamma_{21}} \tag{32}$$

Now, *you'd think* that when $n_{coll} \gg n_{crit}$, i.e. when the downward collisional rate $\gg$ the spontaneous photon emission rate, you'd always have $T_x = T_K$. And this is, in fact, true for the anti-RJ limit $\frac{T_{21}}{T_K} \gg 1$ (in which case we *also* have $\frac{T_{21}}{T_{RJ,K}} \gg 1$) so its inverse can be neglected in the above equation. But owing to the exponentials, this *isn't* true for the RJ limit: there, $T_{RJ,K} \approx T_K$ so $\frac{kT_{RJ,K}}{h\nu} \gg 1$ and then what matters is the product $\frac{kT_{RJ,K}}{h\nu}\frac{n_{crit}}{n_{coll}}$: in effect, n_{crit} gets raised by the large factor $\frac{kT_K}{h\nu}$!

For completeness, we give the relationship in terms of $\frac{n_2}{n_{tot}}$ instead of $\frac{n_2}{n_1}$:

$$\frac{n_2}{n_{tot}} = \frac{\frac{n_2}{n_1}}{1+\frac{n_2}{n_1}} = \frac{g_2}{g_1}\frac{e^{-T_{21}/T_x}}{1+\frac{g_2}{g_1}e^{-T_{21}/T_x}} = \frac{g_2}{g_1}\frac{e^{-T_{21}/T_K}}{1+\frac{g_2}{g_1}e^{-T_{21}/T_K}+\frac{n_{crit}}{n_{coll}}} \tag{33a}$$

$$\frac{n_2}{n_{tot}} = \frac{g_2}{g_1}\frac{1}{\left(1+\frac{n_{crit}}{n_{coll}}\right)\left(1+\frac{T_{21}}{T_{RJ,K}}\right)+\frac{g_2}{g_1}} \tag{33b}$$

6.2.2. *IMPORTANT RESTRICTION ON THIS SOLUTION!!!*

Knowing T_x, and knowing that it is constant within the region, the solution of the equation of transfer is just equation 28

$$T_{RJ,B} = T_{RJ,x}(1-e^{-\tau_\nu}) + T_{RJ,B,BKGND}e^{-\tau_\nu} \tag{34}$$

In section 2.1, we found that if $T_{RJ,B}$ becomes comparable to T_{21}, then the induced and spontaneous radiative rates are nearly equal. We neglected radiative rates in deriving T_x. Thus, *the above solution is only valid if* $T_{RJ,B} \ll T_{21}$. In the optically thick limit, $T_{RJ,B} \rightarrow T_{RJ,K}$. Thus, for $\tau \gtrsim 1$, we require $T_{RJ,K} \lesssim T_{21}$. But usually we are interested in cases where collisional excitation is effective, which in turn means $T_K \gtrsim T_{21}$. Thus, for most cases of interest, the above solution is valid only for $\tau \ll 1$.

6.2.3. *Solution for $\tau \ll 1$*

For $\tau \ll 1$, the background is almost unattenuated so the emergent line brightness, minus the background, is

$$T_{RJ,B,APP} = T_{RJ,B} - T_{RJ,B,BC} \rightarrow (T_{RJ,x} - T_{RJ,B,BC})\tau_\nu \tag{35}$$

τ depends on T_x: using equation 21 and letting $T_{LC,APP}$ be the apparent line-center radiation temperature, we have

$$T_{RJ,LC,APP} = T_{RJ,B,LC} - T_{RJ,B,BC} = \frac{N(H)}{N(H)_{ref}\delta V_{FWHM}} \left(1 - \frac{T_{R,BC}}{T_{RJ,x}}\right) \frac{T_{21}}{\frac{g_2}{g_1} + 1 + \frac{T_{21}}{T_{RJ,x}}} \tag{36}$$

Note that $T_{RJ,x}$ appears in the denominators, which makes the equation look a lot simpler than it really is! Anyway, we have two interesting limits.

First is the *high-density limit* $\frac{n_{crit}}{n_{coll}} \ll 1$. In this case collisions dominate: $T_{RJ,x} \to T_{RJ,K}$ and the complicating $T_{RJ,x}$ in the denominator disappears so we get

$$T_{RJ,LC,APP} = T_{RJ,B,LC} - T_{RJ,B,BC} = \frac{N(H)}{N(H)_{ref}\delta V_{FWHM}} \left(1 - \frac{T_{B,BC}}{T_{RJ,K}}\right) \frac{T_{21}}{\frac{g_2}{g_1} + 1 + \frac{T_{21}}{T_{RJ,K}}} \tag{37}$$

The emergent line intensity $\propto$ column density. This is hardly unexpected, because this is the same case as the 21-cm line treated above.

Next is, of course, the *low-density limit*. With $\frac{n_{crit}}{n_{coll}} \gg 1$, collisions are infrequent and from equation 31b we get:

$$T_{RJ,x} \to \frac{T_{RJ,K}}{1 + \frac{T_{RJ,K}}{T_{21}}} \frac{n_{coll}}{n_{crit}} \tag{38}$$

so $T_{RJ,x} \ll T_{RJ,K}$. If, also, we achieve the perhaps more restrictive condition $\frac{T_{21}}{T_{RJ,x}} \gg (1 + \frac{g_2}{g_2})$, then equation 36 yields

$$T_{RJ,LC,APP} = T_{RJ,B,LC} - T_{RJ,B,BC} \to \frac{N(H)}{N(H)_{ref}} \left(\frac{T_{RJ,K}}{1 + \frac{T_{RJ,K}}{T_{21}}} \frac{n_{coll}}{n_{crit}} - T_{RJ,B,BC} \right) \tag{39}$$

so $T_{RJ,x}$ drops out. If $T_{R,BC} \to 0$, then $T_{R,LC} \propto Nn$, which is equivalent to the *emission measure* dependence for the low-density limit in HII regions. Fundamentally, the reason is obvious: the optical depth is small, so every collisional excitation leads to a photon and it escapes without hindrance because $\tau \ll 1$.

7. MORE GENERAL CASE OF KNOWN T_K AND n: ESCAPE PROBABILITY FORMALISM

But what is T_x in the general case? A complicated question, so complicated that it can't be done analytically except by approximation. More generally, we have to include radiative transitions so instead of equation 30 we have

$$n_1(n_{coll}\gamma_{12} + JB_{12}) = n_2(n_{coll}\gamma_{21} + A_{21} + JB_{21}) \tag{40}$$

So the population ratio, i.e. T_x, depends on J (equivalent to T_B), T_K, and n_{coll}. We know T_K and n_{coll}. (In fact, most generally we know the emergent intensities from measurement and want to determine T_K and n_{coll}! But for now we'll regard density and kinetic temperature as given.)

But we don't know T_B. Knowing T_B requires solving the equation of transfer. But to solve that, we need to know T_x and τ' (i.e., absorption coefficient) as functions of depth within the region. But they, in turn, depend on T_B if radiative excitations are important. They are all coupled—a difficult problem! This general situation is often called *photon trapping*, *radiative trapping*, *line trapping*... It is not amenable to a straightforward analytic solution.

For this reason, we introduce an approximation called the *escape probability formalism.* We will illustrate all this with the simplifying example of a two level system.

7.1. Equation of Statistical Equilibrium for the Photons

Here we consider the photons as a gas. Suppose we are deep in the middle of a slab and $\tau > 1$, so photons have a hard time escaping. Photons are created in only two ways: by spontaneous emission (n_2A_{21}) and induced emission (n_2JB_{21}). They are removed in only two ways: by absorption (n_1JB_{12}) and by escaping the immediate region (as opposed to the entire slab). Here, the rates are cm^{-3} sec^{-1}.

We consider escape to occur only for those photons that are spontaneously emitted. If a photon produced an absorption before escaping, we consider it to be part of the photon gas. In this way, the rate of escape from the region is $\beta n_2 A_{21}$, where β is the *escape probability.* Setting the production rate equal to the loss rate, we get

$$n_2(A_{21} + JB_{21}) = n_1JB_{21} + n_2A_{21}\beta \tag{41a}$$

$$(1-\beta)n_2A_{21} = J(n_1B_{12} - n_2B_{21}) \tag{41b}$$

We have neglected photons from the external radiation incident on the slab. External radiation is partially absorbed on its way into the cloud; one could include this contribution as an additive term above.

7.2. Equation of Statistical Equilibrium for the Level Population and the Excitation Temperature inside the Slab

Inside the slab, the excitation temperature T_x (equivalently the population ratio $\frac{n_2}{n_1}$) is determined by the usual equation of statistical equilibrium with no terms omitted (equation 40):

$$n_1(n_{coll}\gamma_{12} + JB_{12}) = n_2(A_{21} + n_{coll}\gamma_{21} + JB_{21}) \tag{42}$$

which can't be solved without knowing the mean intensity J. But J is given by equation 41. Eliminating J above, we get

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-T_{21}/T_x} = \frac{g_2}{g_1} \frac{e^{-T_{21}/T_K}}{1 + \frac{\beta n_{crit}}{n_{coll}}} = \frac{g_2}{g_1} \frac{1}{\left(1 + \frac{\beta n_{crit}}{n_{coll}}\right)\left(1 + \frac{T_{21}}{T_{RJ,K}}\right)} \tag{43a}$$

$$T_{RJ,x} = \frac{T_{RJ,K}}{1 + \frac{\beta n_{crit}}{n_{coll}}\left(1 + \frac{T_{RJ,K}}{T_{21}}\right)} \tag{43b}$$

(HM equation 5.26; SS equation 5)[1]. Compare this with equation 31, which is the equivalent for no radiative trapping, and we see that n_{crit} is lowered by the factor β. The reason is simple: without trapping, $n_{crit} = \frac{A_{21}}{\gamma_{21}}$ is defined by a competition between spontaneous photon emission and collisional depopulation; with radiative trapping, photons increase n_2, which is equivalent to decreasing A_{21}.

For completeness, we give the relationship in terms of $\frac{n_2}{n_{tot}}$ instead of $\frac{n_2}{n_1}$:

$$\frac{n_2}{n_{tot}} = \frac{g_2}{g_1} \frac{e^{-T_{21}/T_x}}{1 + \frac{g_2}{g_1} e^{-T_{21}/T_x}} = \frac{g_2}{g_1} \frac{e^{-T_{21}/T_K}}{1 + \frac{g_2}{g_1} e^{-T_{21}/T_K} + \frac{\beta n_{crit}}{n_{coll}}} \tag{44a}$$

$$\frac{n_2}{n_{tot}} = \frac{g_2}{g_1} \frac{1}{\left(1 + \frac{\beta n_{crit}}{n_{coll}}\right)\left(1 + \frac{T_{21}}{T_{RJ,K}}\right) + \frac{g_2}{g_1}} \tag{44b}$$

7.3. The Emergent Intensity

We make the approximation that all physical conditions, including β, are independent of position. This means that T_x is also independent of position. Then the emergent intensity is given by equation 28

[1] Note we have neglected the external continuum in equation 43. This means that, as $n_{coll} \rightarrow 0$, this equation incorrectly yields $T_{RJ,x} \rightarrow 0$; in fact, we expect $T_x \rightarrow T_{R,BKGND}$ in this limit.

$$T_{RJ,B,LC} = T_{RJ,x}(1 - e^{-\tau_{LC}}) + T_{RJ,B,BKGND}e^{-\tau_{LC}} \tag{45a}$$

$$T_{RJ,LC,APP} = T_{RJ,B,LC} - T_{RJ,B,BC} = (T_{RJ,x} - T_{RJ,B,BC})(1 - e^{-\tau_{LC}}) \tag{45b}$$

(SS equation 4), where $T_{RJ,x}$ is from equation 43 and τ_{LC} from equation 21; you can solve these numerically, as in our example below.

The behavior is the same as discussed in §6.2, with two exceptions: First, the restriction $T_{RJ,B} \ll T_{21}$ doesn't apply; second, βn_{crit} replaces n_{crit}. In particular, for $\tau_{LC} \ll 1$ and $\frac{\beta n_{crit}}{n_{coll}} \ll 1$ there is an emission-measure like dependence, with the emergent intensity $\propto N_{tot} n_{coll}$.

7.4. The cooling rate

In the optically thin case, the cooling rate $n^2\Lambda = n_2 A_{21}$; all spontaneously emitted photons leave. Here, only a fraction β leave, so $n^2\Lambda = \beta n_2 A_{21}$.

7.5. The Escape Probability for Various Geometries

The escape probability depends on the optical depth and the geometry. An excellent summary is in the RADEX manual, but it is sometimes unclear, e.g. for a sphere, whether stheir definition of 'optical depth' is for the diameter or for the radius. Below we gather some expressions from different sources, not all of which agree. In principle, dust can contribute to β, but for FIR lines it is usually negligible (see section 5.1).

7.5.1. A Uniform Slab of width $2\tau_{LC}$

For a slab of total width $2\tau_{LC}$, a good enough approximation[2] is

$$\beta = \frac{1 - e^{-4.1\tau_{LC}}}{4.1\tau_{LC}} \tag{46}$$

In contrast, the RADEX manual gives

$$\beta = \frac{1 - e^{-3\tau_{LC}}}{3\tau_{LC}} \tag{47}$$

[2]See refs quoted by Genzel or HM. HM provide a better analytical form for this approximation, but its answers are reasonably close to equation 46, which is more commonly used.

Who knows which is right. In any case, the emergent intensity for a slab depends on the viewing angle. Using this equation with equation 45 gives the brightness normal to the slab.

7.5.2. *A Uniform Sphere of center-to-edge τ_{LC}*

Averaging over a uniform sphere of radial opticcal depth τ_{LC}, a good enough approximation (according to Draine, equation 19.11) to a more complicated expression (Ferland & Osterbrock Equation (4.46); also the RADEX manual) is

$$\langle\beta\rangle_{cloud} = \frac{1}{0.5\tau_{LC}} \tag{48}$$

However, the equations don't match as well as Drain implies, so there's something wrong here.

7.5.3. *The Sobolev Approximation for an expanding slab*

In an expanding region, the central frequency in the line shape function is a linear function of position. Thus, the optical depth for a photon depends on the velocity gradient; once the central frequency moves by $\sim$ the line width, the photon suffers no more opacity. From SS, define

$$\tau_{SOB} = \frac{\lambda^3 A_{21} n_{tot}}{8\pi \frac{dV}{dz}} \frac{g_2}{g_1} \frac{\frac{T_{21}}{T_{RJ,x}}}{1 + \frac{g_2}{g_1} + \frac{t_{21}}{T_{RJ,x}}} \tag{49a}$$

which can be expressed in terms of equation 21 by setting $N(H) = n(H)/\frac{\delta V_{FWHM}}{L_{SOB}}$, where $\frac{\delta V_{FWHM}}{L_{SOB}} = \frac{1}{1.065}\frac{dV}{dz}$.

$$\tau_{SOB} = \frac{1.065\frac{n(H)}{dV/dz}}{N(H)_{ref}} \frac{\frac{T_{21}}{T_{RJ,x}}}{\frac{T_{21}}{T_{RJ,x}} + \frac{g_2}{g_1} + 1} \tag{49b}$$

With this, the escape probability is given by equation 46, but with the factor 4.1 replaced by 1 (Draine (19.28); RADEX manual);

$$\beta = \frac{1 - e^{-\tau_{SOB}}}{\tau_{SOB}} \tag{50}$$

7.6. A Numerical Example

Using equations 21, 43, and 45, you can't obtain a closed-form expression for the physical conditions and emergent intensity. But you can use them numerically to make a graph. Figure 2 exhibits some solutions for the CII 158 μm line. Here, we took $T_{RJ,B,BG} = 0$ K, $T_K = 50$ K, and three values of $\tau_{LC,(T_x=0)} = \frac{N(H)}{N(H)_{ref}\delta V_{FWHM}}$, which in the figure we denote $NRATIO$; the three values are $NRATIO = (0.2, 1, 5)$ as shown.

Instead of doing this yourself, you can use the widely-used RADEX program, which is available on-line at the LAMBDA site.

7.7. Comment for Aficionados

Aficionados of radiative transfer realize that there are two limiting approximations in treating photon scattering. Here, by "photon scattering", we mean the process in which a photon is absorbed in a $1 \rightarrow 2$ transition and then subsequently re-emitted in a $2 \rightarrow 1$ transition.

One approximation is *coherent scattering*, in which the outgoing photon has exactly the same frequency as the incoming one. The other is *noncoherent scattering* or *complete redistribution*, which means that the scattered photon comes off with absolutely no relationship to the incoming one—in short, that the scattered photons have line shape is ϕ_ν. An accessible discussion of the differences is given by Ambartsumian (1958, Theoretical Astrophysics, chapters 3.5 and 14.2). It is said that if the line shape function ϕ_ν is defined by thermal Doppler shifts, then complete redistribution is a good approximation. That's good: it means that all processes, including emission and scattering, are governed by the same line shape function ϕ_ν. That assumption is implicit in the escape probability treatment. The other extreme, where photons are in the damping wings, would be better approximated by coherent scattering.

SOLID, DOT, DASHED HAVE NRATIO = .2, 1, 5

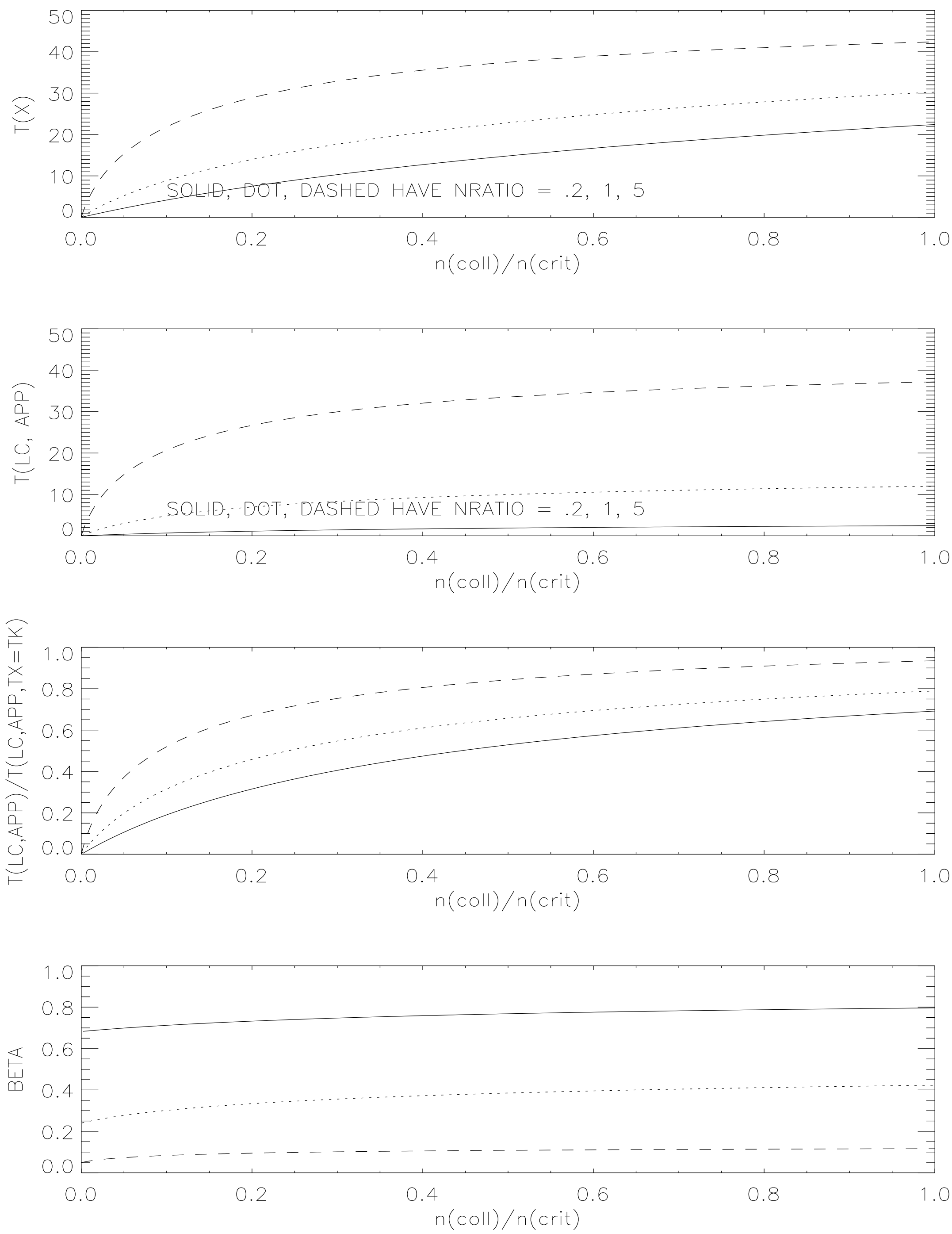

Fig. 2.— Graphs of excitation temperature, emergent radiation temperature, and beta for the CII line