ESSENTIALS OF WEEK 2, ASTRONOMY 127

3. More Non-LTE:

3.1. Radiative transitions and rates. In week 1 we covered collisions and derived a relation between collisional rates using thermodynamical arguments. Here we follow exactly the same procedure for radiative rates. For upwards radiative transitions there is a single process, absorption of a photon; the rate is

$$R_{12}^{rad} = \bar{J}_\nu B_{12} \tag{10}$$

where B_{12} is the Einstein B coefficient for the upwards transition and $\bar{J}_\nu$ is the *mean intensity*, which is the direction-averaged specific intensity:

$$\bar{J}_\nu = \frac{\int \bar{I}_\nu d\Omega}{4\pi}(11) \tag{11}$$

In these equations, the "bar" sign over J_ν and I_ν means that the quantity is averaged in frequency over the width of the spectral line. For downwards transitions we have the analogous photon-induced rate $R_{21} = \bar{J}_\nu B_{21}$ and, in addition, the spontaneous rate, which is equal to the Einstein A_{21}.

3.1.1. Specific intensity, mean intensity, and flux: Note the difference between *specific intensity* I_ν, *mean intensity* J_ν, and *flux*. The specific intensity I_ν measures the intensity of radiation as a function of direction and has units erg/sec-cm^2-Hz-ster. Mean intensity J_ν is the average of I_ν over solid angle and so is a measure of the total number of photons coming in, *independent* of the *direction*. The units of mean intensity are erg/sec-cm-Hz^2—the "per ster" is gone because it is an average over all directions. This quantity—the total number of photon coming in—is what is relevant for photon-induced transitions, because the total rate at which an atom suffers such transitions is proportional to the total number of photons coming in, and it doesn't matter what direction they come from. In contrast, *flux* is the net power per cm^2 flowing across a surface, and is equal to $\int I \cos\theta d\Omega$, where θ is measured with respect to the direction of interest. Flux is the quantity normally dealt with in elementary astronomy, and certainly *does* depend on direction. For example, the luminosity of a star is the flux at the stellar surface multiplied by the star's surface area. Thus the units of flux are erg/sec-cm^2. To appreciate the difference between flux and mean intensity, imagine yourself suspended in the middle of a blackbody cavity at temperature T. The *flux* is *zero*: there is as much radiation flowing in one direction as in the opposite direction. But the *mean intensity* is equal to $B_\nu(T)$, the blackbody radiation field, and if T is hot enough you will roast—evenly on all sides.

3.1.2. Induced and spontaneous radiative rates. If radiative rates dominate collisional rates by a large factor, *and* if the radiation field is a *true* blackbody (a rare condition in astronomy!) at temperature T_r, then

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} \exp(-E_{21}/kT_r)(12) \tag{12}$$

This is known as the “principle of detailed balancing”, and arises from fundamental considerations in thermodynamics and statistical mechanics. This fact allows us to write

$$\frac{R_{12}^{rad}(T_r)}{R_{21}^{rad}(T_r)} = \frac{g_2}{g_1} \exp(-E_{21}/kT_r)(13) \tag{13}$$

which is a relation between the upwards and downwards radiative rates. In term of the Einstein coefficients, this relation is

$$\frac{B_{12}}{B_{21} + A_{21}/B_\nu(T_r)} = \frac{g_2}{g_1} \exp(-E_{21}/kT_r)(14) \tag{14}$$

and this leads to relationships among the Einstein coefficients. This relations are, fundamentally, among atomic properties, and thus hold universally. They are

$$\frac{B_{12}}{B_{21}} = \frac{g_2}{g_1}(15) \tag{15}$$

$$B_{21} = \frac{A_{21}c^2}{2h\nu^3}(16) \tag{16}$$

The ratio of downward-induced (“maser”-type transitions) to spontaneous rates is

$$\frac{\text{induced}}{\text{spontaneous}} = \frac{\bar{J}_\nu B_{21}}{A_{21}}$$

which reduces to

$$\frac{\text{induced}}{\text{spontaneous}} = \frac{1}{\exp(h\nu/kT) - 1}$$

which shows that spontaneous transitions dominate induced transition for $h\nu \gtrsim kT$. This, in turn, means that in radio astronomy (usually characterized by $h\nu \ll kT$) induced transitions are *very important*. For optical/UV lines in interstellar space, the mean intensity J is reduced far below the blackbody value by the geometrical dilution, so in most cases we can neglect induced transitions when calculating transition rates.

In general, the Einstein A is given by

$$A_{21} = \frac{64\pi^4\nu^3}{3c^3h}|\mu|^2(17) \qquad (17)$$

or

$$A_{21} = 1.75 \times 10^{44} \left(\frac{\nu}{2.47 \times 10^{15}\mathrm{Hz}}\right)^3 |\mu|^2 \ \ \mathrm{sec}^{-1}(18) \qquad (18)$$

where we have normalized the frequency ν to that for the Lyman α ($n = 2 - 1$) transition in the H atom. $|\mu|$ is called the "matrix element for the transition". For dipole transitions it is also called the "dipole moment" and is equal to

$$\mu_{12} = e\int \Psi_1^*\mathbf{r}\Psi_2 dV \sim er_{Bohr}(19) \qquad (19)$$

or, in classical terms, the electron charge times the mean separation of the electron and proton. With $e = 4.8\times10^{-10}$ (esu–use this value in all formulae given in this course) and $r_{Bohr} = 0.53\times10^{-8}$ cm, equation (18) gives $A_{21} = 1.0 \times 10^9$ sec^{-1} for the Lyman α transition. With an accurately-calculated value for $|\mu|$, the correct value for A_{21} is 3.8×10^9 sec^{-1}. Equation (18 + 19) gives typical values for the Einstein coefficients of "allowed" transitions. However, many of the transitions we will deal with are "forbidden". This means that the integral (19) is equal to zero, and $|\mu|$ is given at a higher level of approximation by "electric quadrupole" or "magnetic dipole" transitions. Such rates are smaller by a factor of typically $\sim 10^6$ so have much smaller values of A—often about 1 sec^{-1} for optical transitions.

3.2. Combining collisions and radiation. From equation (2) (see week 1), and neglecting induced radiative rates because J is so small in interstellar space, we get

$$\frac{n_2}{n_1} = \frac{R_{12}^{coll}}{R_{21}^{coll} + A_{21}}(20) \qquad (20)$$

Let us now include, explicitly, the dependency of collisional rates on the density of colliding particles by writing $R = n_{collider}P$; here we take R to be the collision *rate per collidee* (units of R: sec^{-1}) and P to the *probability of a collision per collidee* for [$n_{collider} = 1$ cm^{-3}] (units of P: cm^3 sec^{-1}). We do this because P, which is just $\langle\sigma v\rangle$ in equation (5), is an atomic parameter and depends only on temperature, not density. Then equation (20) becomes

$$\frac{n_2}{n_1} = \frac{P_{12}^{coll}/P_{21}^{coll}}{1 + (A_{21}/n_{collider}P_{21})}(21) \qquad (21)$$

or

$$\frac{n_2}{n_1} = \frac{\frac{g_2}{g_1} \exp(-E_{21}/kT_k)}{1 + (A_{21}/n_{collider}P_{21})}(22) \tag{22}$$

or

$$\frac{n_2}{n_1} = \frac{\text{LTE value for } T_k}{1 + (A_{21}/n_{collider}P_{21})}(23) \tag{23}$$

This makes it explicitly clear that there is just *one single parameter*, $(A_{21}/n_{collider}P_{21})$, *that describes the relative importance of collisional and radiative rates.*

We can imagine two limiting cases: the "high density" case in which $(A_{21}/n_{collider}P_{21}) \ll 1$ and the "low density" case. In the high density case we recover the LTE distribution, the Boltzmann distribution at the kinetic temperature. In the low density case we get

$$\frac{n_2}{n_1} = \frac{n_{collider}P_{21}}{A_{21}} \times (\text{LTE value for } T_k)(24) \tag{24}$$

which is clearly much smaller than the LTE value. Physically, the reason is simple: in LTE, upward collisions are balanced by downward collisions, which leads to the LTE distribution. But in the low density case, the downward radiative rate is very much greater than the collisional rate— there is no time for a downward collision to occur. Each upward collision is followed immediately by a downward radiative transition, so atoms remain in the upper state only a very short time. Consequently, the population of state 2 is very small.