3.3. Photon emission rates. In general, the number of photons emitted per second per cm^{-3} (note: per unit volume, not per emitting particle) by the spontaneous emission process is

$$\frac{d\mathcal{N}}{dt} = n_2 A_{21} \tag{25}$$

To relate this to the total abundance of the atoms, we must express n_2 in terms of n_{tot}, the total volume density of the atoms; $n_{tot} = n_1 + n_2$. We have already calculated the ratio n_2/n_1 and can write

$$n_2 = \frac{n_{tot}}{1 + (n_1/n_2)} \tag{26}$$

Consider $d\mathcal{N}/dt$ for the high- and low-density cases. In the high-density case, n_1/n_2 is given by the Boltzmann distribution so we get

$$\frac{d\mathcal{N}}{dt} = \frac{n_{tot} A_{21}}{1 + \frac{g_1}{g_2}\exp(E_{21}/kT_k)} \qquad \text{(LTE CASE)} \tag{27}$$

Note that $d\mathcal{N}/dt$ is **(1)** *linearly* proportional to the total volume density; **(2)** *proportional to* A_{21}; and, aside from these, depends only on temperature T_k. Because $\mathcal{N}$ is the number per unit volume, the number of photons emitted is proportional to n_{tot}, as we might expect.

The low-density limit is completely different. Here, $n_2/n_1 \ll 1$ so we can take $n_{tot} = n_1$ which gives

$$\frac{d\mathcal{N}}{dt} = n_{tot} n_{collider} P_{12}^{coll} \qquad \text{(LOW – DENSITY LIMIT)} \tag{28}$$

or

$$\frac{d\mathcal{N}}{dt} = n_{tot} n_{collider} P_{21}^{coll} \frac{g_2}{g_1} \exp(-E_{21}/kT_k) \tag{29}$$

As we shall see, in many cases $n_{collider} \propto n_{tot}$. Thus $d\mathcal{N}/dt$ is **(1)** proportional to the *square* of the total volume density; **(2)** *independent* of A_{21} (!); and, aside from these, depends only on temperature T_k. Note that $\mathcal{N}$ is proportional *not* to n_{tot} but to its *square.* Furthermore, the photon emission rate is *independent* of the Einstein A, and this fact may be surprising. However, the interpretation is simple: in the low-density limit, every upwards collisional excitation is followed by a spontaneous transition. Thus *the number of photons emitted is equal to the number of upwards collisional transitions.* In fact, we can write equation (29) as

$$\frac{d\mathcal{N}}{dt} = n_1 n_{collider} P_{12}^{coll} = R_{12}^{coll} \tag{30}$$

4. HII Regions.

4.1. Ionization equilibrium. HII means H^+—ionized hydrogen. We want the ratio of ionized to neutral Hydrogen, n_{HII}/n_{HI}. In LTE, this is given by the Saha equation. We don't have LTE, so instead we must use use statistical equilibrium. Take state 2 as ionized, state 1 as neutral. Then, as before,

$$\frac{n_{HII}}{n_{HI}} = \frac{n_2}{n_1} = \frac{R_{12}}{R_{21}} \tag{31}$$

R_{12} is the ionization rate. There are two possible processes: collisions and radiation. Radiation is stronger by a factor of order 10^{10}; the photons come from the central star. R_{21} is the recombination rate; recombination occurs by the process

$$e + p \rightarrow HI + photon \tag{32}$$

The ionization rate (per H atom) is

$$R_{12} = \int_{\nu(IP)}^{\infty} \frac{4\pi J_\nu}{h\nu} s_\nu d\nu \tag{33}$$

Here the first factor, $\frac{4\pi J_\nu}{h\nu}$, is the number of photons/sec-cm^2; s_ν is the cross section for ionization by a single photon; and the integral is carried out over all photons able to ionize the atom. We can write this in a bit simpler form by approximating the integral:

$$R_{12} = \langle \frac{N_u}{4\pi D^2} \rangle \langle s_\nu \rangle \tag{34a}$$

where N_u is the total number of ionizing photons emitted per second from the star, D is the distance from the star, and the symbols $\langle\rangle$ denote averages over frequency. For HI, the cross section is roughly

$$s_\nu = 6.5 \times 10^{-18} \left(\frac{\nu(IP)}{\nu} \right)^3 \ \ \mathrm{cm}^2 \tag{34b}$$

This varies rapidly with frequency, so the $\langle\rangle$ process will be just an approximation. For even the hottest stars we are in the Wein portion of the blackbody curve, so there aren't very many photons with energies that vastly exceed the ionization potential of hydrogen; so let's take $\langle(\nu(IP)/\nu)^3 = 1/6.5$ for convenience so that $\langle s_\nu \rangle = 1.0 \times 10^{-18}$ cm^2. Values for N_u can be found in the handout. For an O5 star, $N_u \sim 10^{50}$ sec^{-1}. A typical distance is 1 pc $= 3.1 \times 10^{18}$ cm. This gives, for a typical ionization rate, $R_{12} \sim 10^{-6}$ sec $^{-1}$. It takes about 10^6 seconds—about 1 week—for a typical H atom in an HII region to be ionized.

The recombination rate per HII ion (proton) is

$$R_{21} = n_e \langle \sigma v_{th} \rangle \tag{34c}$$

where $\langle \sigma v_{th} \rangle \approx 2.7 \times 10^{-13} T_4^{-0.7}$ cm^3 sec^{-1} is the usual average of cross section over velocity (see equation (9), week 1). (T_4 is temperature in units of 10^4 K). In an HII region, most of the electrons come from ionization of HI, so that $n_e = n_{HII}$. Typically, $T_4 = 1$ so $R_{21} \approx n_e 2.7 \times 10^{-13}$ s^{-1}. For a typical HII region $n_e \sim 10^3$ cm^{-3}, so $R_{21} \sim 3 \times 10^{-10}$: it takes about 100 years for a proton to recombine with an electron. This makes the degree of ionization very high!

$$\frac{n_{HI}}{n_{HII}} = \frac{R_{21}}{R_{12}} \sim \frac{3 \times 10^{-10}}{10^{-6}} \approx 3 \times 10^{-4} \tag{35}$$

4.2. Sizes of HII Regions. To get the size, we can apply the concept of statistical equilibrium not to just a single cm^3 but, rather, to the whole HII region:

$$\text{Recombination rate in whole HII region} = \text{Ionization rate in whole HII region} \tag{36}$$

The recombination rate in the whole HII region is just the rate per unit volume multiplied by the HII region volume. The recombination rate *per unit volume* is from equation (34c and following text [or equation (9), week 1]), multiplied by n_{HII} (which is equal to n_e):

$$\text{Recombination rate in whole HII region} = \frac{4\pi R_{HII}^3}{3} n_e^2 R_{21} \approx \frac{4\pi R_{HII}^3}{3} n_e^2 2.7 \times 10^{-13} T_4^{-0.7} \text{ sec}^{-1} \tag{37}$$

The ionization rate in the whole HII region is just N_u (!). Thus the size is just given by solving

$$N_u = \frac{4\pi R_{HII}^3}{3} n_e^2 2.7 \times 10^{-13} T_4^{-0.7} \tag{38}$$

Note that $R_{HII} \propto N_u^{1/3} n_e^{-2/3}$.

4.3. Intensity of recombination lines and IR emission. When the atoms recombine they emit photons as the electrons descend own the energy level ladder. Transitions down to level 1 emit UV photons and are called the "Lyman series" lines. Transitions down to level 2 emit optical photons and are called "H lines"; the 3-2 line is called the Hα line and the 4-2 line the Hβ line, for example.

Transitions down to level 3 emit in the infrared. Astronomers observe all of these lines, and many more.

To calculate the intensity of the Hα line we simply need to know the fraction of recombinations that end up yielding this transition. For Hα this fraction is just about 0.5. For Hβ the fraction is about 0.11.

The Lyman series lines are more complicated. First, realize that even though the HII region is highly ionized, it does nevertheless contain some HI atoms. And the lion's share of these are in the $n = 1$ state, the ground state. This makes the optical depth of the Lyman series lines enormous. A Lyman series photon cannot move very far before exciting another atom up to the $n = 2$ state. The photon is trapped inside the HII region. Eventually all electrons end up in the $n = 2$ state, trying to emit a Lα photon ($n = 2 - 1$). Such photons cannot leave the HII region. They suffer one of two fates: about 30% end up in the "two-photon" process, in which a virtual, temporary quantum state appears whose energy lies between those of the $n = 1$ and $n = 2$ states; the electron makes a transition from the $n = 2$ state to this virtual state, then from the virtual state to the $n = 1$ state, emitting two photons in the process. The remaining $\sim 70\%$ of the Lα photons hit dust grains, which heat up and then emit as solid bodies in the infrared. This makes HII regions very bright infrared emitters. The infrared power emitted by these grains is:

$$\text{IR luminosity} \approx 0.7 \times \text{Recombination rate in whole HII region} \times \text{Energy of L}\alpha\text{ photon} \tag{39a}$$

or

$$\text{IR luminosity} \approx 0.7 \times N_u \times 10.2 \text{ ev sec}^{-1} \tag{39b}$$

4.4. Temperatures of HII regions. To get temperature, we equate heat input (denoted $n\Gamma$; units erg cm^{-3} sec^{-1}) and heat output (denoted $n^2\Lambda$; same units). (*Note:* These rates are *per unit volume.* Heat input normally comes from the interaction between an externally-generated photon field and the particles, so is proportional to the *first* power of volume density; heat output normally arises from two-body collisional processes, so is proportional to the *second* power of volume density. This explains the convention used here involving the differing powers of volume density for heating and cooling. *However,* in many circumstances these rates have different dependencies on volume densities. Examples include heating in HII regions, heating by mechanical means such as shocks, and cooling in optically thick media.) Both heating and cooling depend on T.

$$n\Gamma(T) = n^2\Lambda(T) \tag{40}$$

Energy gain comes from the kinetic energy of electrons injected by the photoionization process.

It is like equation (33) (the ionization rate) times the energy injected per ionization (which is just the excess of energy over that required to just ionize the atom):

$$n\Gamma(T_k) = n_{HI} \int_{\nu(IP)}^{\infty} \frac{4\pi J_\nu}{h\nu} s_\nu (h\nu - h\nu(IP)) d\nu \tag{41}$$

and we can replace the integral with frequency-averaged values, as in equation (34a):

$$n\Gamma(T_k) = n_{HI} \langle \frac{N_u}{4\pi D^2} \rangle \langle s \rangle \langle (h\nu - h\nu(IP)) \rangle = n_{HI} R_{12} \langle (h\nu - h\nu(IP)) \rangle \tag{42}$$

But we realize that, from ionization equilibrium, $n_{HI}R_{12}$ is the total recombination rate *per unit volume*, which is equal to the total ionization rate per unit volume, so

$$n\Gamma(T_k) = \text{Recombination rate per volume} \times \langle (h\nu - h\nu(IP)) \rangle \tag{43}$$

Realize that the factor $\langle (h\nu - h\nu(IP)) \rangle$ depends only on the temperature of the star T_*. In fact, an approximate calculation gives $\langle (h\nu - h\nu(IP)) \rangle \approx 1.4kT_*$. Thus we can write (from equation (34c) and the following text)

$$n\Gamma(T_k) \approx n_e^2 3.8 \times 10^{-13} T_4^{-0.7} kT_* \tag{44}$$

Energy loss comes from two processes: one, when an electron recombines with a proton, its kinetic energy is removed from the gas; and two, the free electrons collide with protons and emit "free-free" radiation. In the first process the power loss per unit volume is (recombination rate) $\times \frac{3}{2} kT_k$ (each recombination removes the average thermally energy [actually, it removes a bit less because the recombination cross section increases with decreasing electron velocity]). The free-free emission process roughly doubles the power loss rate. Together, the two loss processes give

$$\Lambda(T_k) \approx n_e^2 2.9 \times 10^{-13} T_4^{-0.7} kT_k \tag{45}$$

Equating (44) and (45) gives

$$T_k \approx 1.3 T_* \tag{46}$$

This answer is *wrong!* Not because we've goofed; rather, because it doesn't match observations. It predicts that HII regions have temperatures comparable to the surface temperatures of the central stars. In fact, though, their temperatures are much lower, typically 8000 K.

4.5. Cooling by heavy elements. We have included cooling by H, but have neglected other cooling processes. More important than H cooling is heavy element cooling. To understand this we need to understand the energy level diagrams of heavy elements.

4.5.1. Spectroscopy of multielectron atoms.

Simplest example: the H-like atoms with one electron in p shell. Example: Li. Lowest n for p shell is $n = 2$ ($n = 1$ is taken by the two electrons that fill the s shell). Energy depends not only on n but also on l.

With > 1 electron in p shell we must vectorially add the l and s of the p-shell electrons to get the total angular momentum L and the spin S. $\bar{L} = \Sigma \bar{l}$; $\bar{S} = \Sigma \bar{s}$. The total angular momentum $\bar{J} = \bar{L} + \bar{S}$. Thus for single values of L and S we get several J's: $J = L + S, L + S - 1, \ldots, L - S$.

For L and S constant, the different energies for the various J's are closely spaced and correspond to IR wavelengths. The small splittings are known as "fine structure" and a single set of such closely-spaced levels is known as a "term." If levels have different values of L and S, then the energy differences usually correspond to optical or UV wavelengths. The complete spectroscopic designation of a state involves specifying L, S, and J. L is normally designated by a letter: S corresponds to $L = 0$, P to 1, D to 2. The spectroscopic designation is conventionally written (example for $L = 1$)

$$^{2S+1}P_J \tag{47}$$

The *degeneracy* or *statistical weight* of a level is equal to $g_J = 2J + 1$. The statistical weight of a term is the total number of sublevels, i.e. the sum of g_J over all values of J in the term.

For electron configurations of the form $1s^2 2s^2 2p^x$, we can have only $x \leq 6$. Energy level diagrams for $x = 1$ and 5 are similar to each other; and for $x = 2$ and 4 are similar to each other. Thus there are only three types of energy level diagram. One is exemplified by the H-like atom Li. The other two are exemplified by OIII and OII (O^{++} and O^{+}: $x = 3$ and 4, respectively). As we shall see, $x = 3$-type ions are useful for measuring temperatures of the emitting gas and $x = 2$-types for measuring densities.

The different types of transitions have different selection rules. There are several selection rules for ordinary "electric dipole" transitions (in other words: selection rules for the integral in equation (19) being nonzero). One of these is

(1): The electron configuration must change by one orbital (i.e., there must be a change in the quantum number n).

None of the heavy-element transitions we will discuss satisfy this rule. Thus *all* of these transitions are *forbidden.* There are other selection rules, and the Einstein A's of the various transitions decrease with the number of selection rules that are violated. The selection rules are

(2): $\Delta J = 0, \pm 1$ (with the exception that $J = 0 \rightarrow 0$ is strictly forbidden).

(3): $\Delta S = 0$.

(4): $\Delta L = 0, \pm 1$ (and also required: $\Delta l = \pm 1$, i.e. L changes because the angular momentum of one of the electrons changes, not because the individual l's remain constant but add up in a different vectorial way).

It is also worth recalling (equation 17) that $A \propto \nu^3$, so that low-frequency transitions are intrinsically less probable than high-frequency ones.

4.5.2. Forbidden lines as densitometers and thermometers. (See text, pages 109-115). OIII has three terms spaced by energies of order 2 ev (see text, Figure 3.5). In ascending order these are the ^{3}P, the ^{1}D, and the ^{1}S terms (the first consists of three levels, the latter two of one level each). Call these levels 1, 2, and 3 respectively. The collisional rate $R_{12} \propto \exp -(E_{12}/kT_k) \approx \exp -(2.9/T_4)$; the collisional rate $R_{13} \propto \exp -(E_{21} + E_{32})/kT_k) = \exp -(E_{31}/kT_k) = \exp -(6.2/T_4)$. For $T_4 \sim 1$, R_{31} is *very* much more temperature sensitive than R_{21}. Thus the ratio of line intensities I_{32}/I_{21} varies very strongly with temperature. This makes this ratio a good temperature indicator. *Note:* this is a good thermometer only in the low-density limit!

For OII, the ground state is a ^{4}S$_{3/2}$, a single level; the next higher term is a ^{2}D term with two levels that are closely spaced (see text, Figure 3.6). Thus there are two closely-spaced lines, which are in the optical. In the low-density limit, the ratio of line intensities is the ratio of upwards collision rates, which is $g_3/g_2 = 2/3$. In the high-density limit the levels are populated according to LTE at temperature T_k and the ratio of line intensities is $n_3A_{31}/n_2A_{21} = 3.33$; temperature doesn't enter because the levels are so close together that $E/kT_k \ll 1$. Thus the ratio of line intensities depends on density. The range of density to which the intensity ratio is sensitive is $n_e \sim 10^2$ to 10^4 cm^{-3}.

4.5.3. Temperatures of HII regions with heavy element cooling. To compute the cooling $\Lambda(T_k)$ in the low-density limit, we multiply the collisional excitation rate for electron-ion collisions from equation (8) by the energy of the transition, because this is the amount of energy that leaves the system when the photon is emitted. Thus for OIII,

$$\Lambda(T_k) = n_{OIII} n_e P_{12} E_{12} \tag{48}$$

Express $n_{OIII} = x\frac{n_{O_{tot}}}{n_e}n_e$—in words, we assume that all O is OIII, take $\frac{n_{O_{tot}}}{n_e}$ equal to the cosmic abundance of O/H (by number of atoms: equal to 6.3×10^{-4}, use the fact that (because the H is almost fully ionized) $n_H = n_e$, and let x be a correction factor (x must be less than one, but in fact it isn't much less than one) to account for the fact that these assumptions are not quite correct. Then we can write

$$\Lambda(T_k) = xn_e^2 6.1 \times 10^{-23} \frac{\exp -(2.9/T_4)}{T_4^{1/2}} \tag{49}$$

Compare this with the heating in equation (44). Note that both depend on the *square* of n_e; this is because we are n the low- density limit for cooling. Equating (49) and (44) gives

$$T_{k,4} \exp -(2.9/T_{k,4}) = \frac{0.0084}{x} T_{*,4} \tag{50}$$

Solving this rather messy equation gives temperatures of order 8000 K ($T_{k,4} \sim 8$). T_k increases, but only very slowly, with T_*. T_k decreases with increasing x (in words: as you increase the abundance of the coolant, the gas temperature drops).

Note: If you were to calculate the intensity of the OIII line, you would find that it is (surprisingly?) *independent* of x. This is because we have assumed that OIII is the only coolant, and because cooling equals heating, the OIII line intensity must equal the heating no matter what the abundance of oxygen. *In real life*, the presence of other cooling lines that are less sensitive to temperature (including the fine structure transitions in the ^{3}P term of OIII itself) makes the OIII optical line intensity *decrease* with *increasing* O abundance! The upshot of all this is that it is not so easy to obtain the abundances of heavy elements from the intensities of their emission lines!