University of California, Berkeley
Physics 221b Spring 2004

Problem Set 11
due Monday May 3, 2004
(incorrect "note" at end of problem 5 removed.)

1. Consider a real Klein Gordon field,

$$\phi(x) = \sum_k \frac{1}{\sqrt{2\omega_k V}}[a(\vec{k})e^{-ik\cdot x} + a^\dagger(\vec{k})e^{ik\cdot x}] \; . \tag{1}$$

Find an (integral) expression for $a(\vec{k})$ in terms of $\phi(x), \dot{\phi}(x)$. Rewrite this integral expression in terms of the inner product for Klein-Gordon fields of the same mass, $(\phi_1, \phi_2) = i\int d^3x(\phi_1\partial_0\phi_2^* - \partial_0\phi_1\phi_2^*)$. Hint: Consider combinations of Fourier transforms of $\phi(x)$ and $\partial_t\phi(x)$.

2. We defined the mode expansion for a complex Klein-Gordon field in class as

$$\phi(x) = \sum_k \frac{1}{\sqrt{2V\omega_k}}(a_k e^{-ik\cdot x} + b_k^\dagger e^{ik\cdot x}) \tag{2}$$

   (a) In terms of these modes, $a^\dagger, a, b^\dagger, b$, calculate the charge corresponding to the 4 current operator S^μ defined in class:

$$S^\mu = -ie(\phi^\dagger\partial^\mu\phi - \partial^\mu\phi^\dagger\phi) \; . \tag{3}$$

   Here e is the charge of the field ϕ and $\partial_\mu S^\mu = 0$. You will get an infinite constant from reordering the operators in your final answer for the charge, which you should discard.

   (b) Show that the charge is conserved (i.e. time independent).

3. For a real scalar field, define the propagator at two different space time positions as

$$[\phi(x), \phi(y)] = \Delta(x-y)$$
$$\Delta(x) = \frac{-i}{(2\pi)^3}\int d^3k \frac{1}{\omega_k}\sin k\cdot x = \frac{1}{(2\pi)^3}\int d^4k\,\delta(k^2-m^2)\mathrm{sign}(k^0)e^{ik\cdot x} \; . \tag{4}$$

Here x and y are 4-vectors, $k\cdot x = k^\mu x_\mu$ and $\omega_k = k^0 = +\sqrt{\vec{k}^2+m^2}$.

(a) Derive the propagator using the field expansion for $\phi(x)$.

(b) Show that $\Delta(x-y)$ vanishes when $(x-y)^2 < 0$, i.e. when the two points aren't causally related. Hint: start with the two positions x and y being at the same time, but different positions and then argue using Lorentz invariance.

4. Consider the operator $\vec{\Sigma}$ defined by $\vec{\Sigma} = (\Sigma^{23}, \Sigma^{31}, \Sigma^{12})$, where $\Sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Using the γ matrix representation from class, we have

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} . \tag{5}$$

(a) Show that the helicity operator, $\vec{\Sigma} \cdot \hat{p}$ commutes with the first quantized Hamiltonian,

$$H_{first} = -i\gamma^j \partial_j + m . \tag{6}$$

The sum variable $j = 1, 2, 3$ and $\hat{p} = \vec{p}/|p|$.

(b) Calculate the expectation value of the helicity operator i.e.

$$\int d^3x \bar{\psi} \vec{\Sigma} \cdot \hat{p} \psi \tag{7}$$

in the state

$$\psi = A \begin{pmatrix} 1 \\ 0 \\ Bp^3 \\ B(p^1 + ip^2) \end{pmatrix} e^{-ip\cdot x} \frac{\sqrt{m}}{\sqrt{V p_0}} = u_1(\vec{p}) e^{-ip\cdot x} \frac{\sqrt{m}}{\sqrt{V p_0}} \tag{8}$$

where $A = \sqrt{\frac{p^0+m}{2m}}$, $B = \frac{1}{p^0+m}$, and $p = (p^0, 0, p_s, p_s)$. Note that $\int d^3x = V$.

5. (Sakurai, 3-9) At some instant of time, say t=0, the normalized wave function for an electron (or positron) is known to be

$$\psi(x)|_{t=0} = \frac{1}{\sqrt{V}} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} e^{ip_3 x_3} \tag{9}$$

where a, b, c, d are independent of the spacetime coordinates and satisfy $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$. Find the probabilities of observing an electron with (i) spin up,(ii) spin down, (iii) a positron with spin up (iv) a positron with spin down.

Please show your work so that the grader can give you credit for effort even if you get the wrong answer. Also, please try to make your work readable!