University of California, Berkeley
Physics 221b Spring 2004

Problem Set 3
due Feb. 20, 2004

1. Consider the operator

$$\begin{aligned} K &= \frac{1}{2m}\sum_i p_i^2 a_p^\dagger a_p \\ &= \frac{1}{2m}\int d^3r \vec{\nabla}_r a^\dagger(\vec{r}) \cdot \vec{\nabla}_r a(\vec{r}) \end{aligned} \quad (1)$$

Find its expectation value in the state

$$|\Psi^{(1)}\rangle = \int d^3y a^\dagger(y)|0\rangle\psi(y) \quad (2)$$

in terms of the functions $\psi(r), \psi^*(r)$. You may assume that

$$[\vec{\nabla}_z a(z), a^\dagger(y)] = \vec{\nabla}_z \delta(z-y) = -\vec{\nabla}_y \delta(z-y) \quad (3)$$

2. Consider non-relativistic, non-interacting, spin-S particles in a large cube of volume V. The creation and annihilation operators $a_{sp}^\dagger$ and a_{sp} refer to plane wave states with momentum $\vec{p}$ and spin s.

   (a) A generic first quantized operator U has matrix elements $\langle s'p'|U|sp\rangle$. What is the corresponding second quantized operator?

   (b) Construct the second quantized operator for the particle density at $\vec{R}$ in terms of the operators $a_{sp}^\dagger$ and a_{sp}. Start with the first quantized operator $\rho(\vec{R}) = \sum_i \delta(\vec{R}-\vec{r}_i)|r_i\rangle\langle r_i| \to \int d^3r_i \delta(\vec{R}-\vec{r}_i)|r_i\rangle\langle r_i|$.

   (c) What are the operators which create and destroy particles at $\vec{r}$? What is the density operator (above) in terms of these operators (rather than the ones depending upon momentum $\vec{p}$)?

3. Consider non-relativistic, non-interacting, spin-5/2 particles of mass m in a large cube of volume V, again with creation and annihilation operators $a_{sp}^\dagger$ and a_{sp} referring to plane wave states with momentum $\vec{p}$ and spin s.

(a) Describe the ground state $|\phi\rangle$ of a large number N of these fermions and compute the Fermi momentum p_F.

(b) What is $\langle\phi|\rho|\phi\rangle$?

(c) Define the one particle density matrix $G = \langle\phi|a^\dagger_{s'}(\vec{r}')a_s(\vec{r})|\phi\rangle$. Calculate $g(x)$ where $G_{ss'}(\vec{r},\vec{r}') \equiv (N/V)\delta_{s's}g(p_F|\vec{r}-\vec{r}'|)$. Sketch $g(x)$ and describe its features.

(d) How does $g(x)$ change if the particles have spin 1/2?

4. Suppose a diatomic molecule is composed of 2 atoms F which are fermionic and in the same spin state. If for low lying states the system can be considered to rotate about the midpoint of the linear bond between the two F's, what is the spectrum of the excitations in units of the frequency $h\nu_0 = \hbar^2/I$, where I is the moment of inertia of the molecule?

5. Show that for a state with 2 spin-1/2 particles the operator

$$\frac{1}{2}\left(1 + \frac{4}{\hbar^2}\vec{S}^{(1)} \cdot \vec{S}^{(2)}\right) \tag{4}$$

exchanges the spins. Note that it is helpful to rewrite

$$\vec{S}^{(1)} \cdot \vec{S}^{(2)} = \frac{1}{2}(\vec{S}_+^{(1)}\vec{S}_-^{(2)} + \vec{S}_-^{(1)}\vec{S}_+^{(2)}) + \vec{S}_z^{(1)}\vec{S}_z^{(2)}\ . \tag{5}$$

Please show your work so that the grader can give you credit for effort even if you get the wrong answer. Also, please try to make your work readable!