University of California, Berkeley
Physics 221b Spring 2004

Problem Set 4
due Feb. 27, 2004

Note that "problem" means "problem" not "exercise"!

1. Consider N spin 1/2 fermions in the ground state in a box of volume V, i.e. with density n and Fermi momentum p_F.

   Consider the 3 fermion operator

   $$a_s^\dagger(r_1)a_s^\dagger(r_2)a_s^\dagger(r_3)a_s(r_3)a_s(r_2)a_s(r_1) \ . \tag{1}$$

   and evaluate it in the fermionic ground state. You can express the final answer in terms of G, where

   $$\begin{aligned} G_{ss'}(r-r') &= \tfrac{1}{V}\textstyle\sum_p e^{-ip\cdot(r-r')}n_{ps}\delta_{ss'} \\ &= \tfrac{n}{2}3\frac{\sin(p_F|r-r'|)-p_f|r-r'|\cos(p_f|r-r'|)}{(p_F|r-r'|)^3} \ . \end{aligned} \tag{2}$$

   Consider the case $r_1 = \infty(orlargerather), r_2 = -r_3 = r$ and $2rp_F$ small and explain the connection to the expectation value of the 2 fermion operator

   $$a_s^\dagger(r_1)a_s^\dagger(r_2)a_s(r_2)a_s(r_1) \ . \tag{3}$$

   Consider the case $r_1 = 0, r_2 = -r_3 = r$, $2rp_F$ small, and explain qualitatively the leading difference in r dependence with the above $r_1 \sim \infty$ limit. You can do this by considering the difference of the two limits.

2. Consider N spin zero bosons in 1 dimension on a line of length L. (measure $\sum_p \to \frac{L}{2\pi}\int dp$). Take a state with density function $n_p = 1, p \le p_0$ and zero otherwise.

   (a) What is the pair correlation function $\langle\phi|a^\dagger(0)a^\dagger(r)a(r)a(0)|\phi\rangle$ in this state?

   (b) If the N free particles in the state with the above density function have a small potential $v(r_1-r_2) = v_0(\frac{(r_1-r_2)}{L})^2$ added to the Hamiltonian, what is the first order correction to the energy (first order in the potential)?

3. Merzbacher *problem* 22.2 (a) and (b) only.

4. Consider the Hamiltonian discussed in class:

$$\mathcal{H} = -\sum_{j\ell} J_{j\ell} \vec{S}_j \cdot \vec{S}_\ell + B \sum_j (S_j)_z \tag{4}$$

Here $\vec{S}_j$ is the spin vector at position j, etc., and we have $(S_j)_z = -s, -s+1, -s+2, ...s-1, s$ acting on our states. $J_{j\ell} = J_{j+q\,\ell+q} > 0$ (periodic one dimensional lattice, all sites equivalent), $B > 0$ and the number of sites n is large. In sums we'll ignore edge effects and treat $n \sim \infty$. We also have that $J_{j\ell} = J_{\ell j}$. The ground state is the configuration with all the spins aligned and pointing downwards, and on this state

$$\mathcal{H}|-S> = (-\sum_{j\ell} J_{j\ell} s^2 \hbar^2 - nBs\hbar)|-S> \tag{5}$$

The lattice is one dimensional, the spins are not (they are 3 dimensional).

We represented the spin operators as bosons,

$$\begin{aligned} S_+ &= \hbar a^\dagger_{1/2} a_{-1/2} \\ S_- &= \hbar a^\dagger_{-1/2} a_{1/2} \\ S_z &= \tfrac{\hbar}{2}(a^\dagger_{1/2} a_{1/2} - a^\dagger_{-1/2} a_{-1/2}) \end{aligned} \tag{6}$$

For any given site we thus use $a_{1/2,j}$ etc, adding a site label (j in this case). In terms of these operators the ground state is

$$|-S> = \prod_i \frac{(a^\dagger_{-1/2,i})^{2S}}{\sqrt{(2S)!}}|0> \tag{7}$$

where $|0>$ is the Fock vacuum.

Consider the state

$$\frac{1}{\sqrt{2\pi}} \sum_j e^{i\kappa j} a^\dagger_{1/2,j} a_{-1/2,j}|-S> = \frac{1}{\sqrt{2\pi}} \sum_j e^{i\kappa j} (S_j)_+|-S> \tag{8}$$

Show that this is an eigenstate of the exact Hamiltonian and calculate the corresponding eigenvalue. It will be helpful to use the facts that

$$\begin{array}{l} \sum_j J_{j\ell} \equiv J \\ \phi(\kappa) = \sum_j J_{j\ell} e^{i\kappa(l-j)} = e^{i\kappa\ell} \sum_j J_{j\ell} e^{-i\kappa j} \end{array} \tag{9}$$

are *both* independent of ℓ. These both come from the translational invariance above.

5. Consider the spin and creation and annihilation operators of the previous problem. Take the spins to lie on a line with all spins equivalent. This is a ground state of the system when $B \neq 0$. When $B = 0$, any configuration with all the spins pointing in the same direction is a ground state, not only this one. The operator $e^{\frac{i}{\hbar}\vec{S}\cdot\vec{\theta}}$ rotates the spins and thus will rotate the spins to another equal energy ground state in the absence of an external magnetic field. Here $\vec{S} = \vec{S}_1 + \vec{S}_2 + \ldots$. The Hamiltonian thus has no preferred direction but the spins, in aligning together, pick one out, a classic example of spontaneous symmetry breaking. We will consider the vector $\vec{\theta}$ to be of the form $(\alpha, \alpha, 0)$. Now make two approximations: the one we made in class, that on states near $|-S>$, $a^\dagger_{-1/2,i} \sim a_{-1/2,i} \sim \sqrt{2s}$, and additionally that we want α very small, $\alpha^2 \sim 0$. We can view this rotated state as an excitation around the original state and write it as a linear combinations of excitations we found in class, the magnon states

$$a^\dagger_\kappa |-S> = \frac{1}{\sqrt{2\pi}} \sum_j a^\dagger_{1/2,j} e^{i\kappa j} |-S> \quad . \tag{10}$$

What magnon(s) appears in this rotated state and (from class) what is the energy of this excitation when $B = 0$? (Note, this should not be very algebraically involved at all.)

Please show your work so that the grader can give you credit for effort even if you get the wrong answer. Also, please try to make your work readable!