University of California, Berkeley
Physics 221b Spring 2004

Problem Set 7
due April 2, 2004

1. In class we defined the BCS creation and annihilation operators b and $b^\dagger$, fermionic combinations of the ordinary free electron creation and annihilation operators a and $a^\dagger$:

$$\begin{array}{ll} b_k = u_k a_k - v_k a^\dagger_{-k} & b^\dagger_k = u_k a^\dagger_k - v_k a_{-k} \\ b_{-k} = u_k a_{-k} + v_k a^\dagger_k & b^\dagger_{-k} = u_k a^\dagger_{-k} + v_k a_k \end{array} \tag{1}$$

with an inverse relation

$$\begin{array}{ll} a_k = u_k b_k + v_k b^\dagger_{-k} & a^\dagger_k = u_k b^\dagger_k + v_k b_{-k} \\ a_{-k} = u_k b_{-k} - v_k b^\dagger_k & a^\dagger_{-k} = u_k b^\dagger_{-k} - v_k b_k \end{array} \tag{2}$$

Here u_k, v_k are real constants. The BCS vacuum $|BCS\rangle$ has no b quasiparticles, i.e.

$$b_k|BCS\rangle = 0 \tag{3}$$

In terms of the Fock vacuum obeying $a_k|0>= 0$,

$$|BCS\rangle = \Pi_{k>0}(u_k + v_k a^\dagger_k a^\dagger_{-k})|0\rangle \ . \tag{4}$$

However, $|0\rangle$ does have b quasiparticles.

   (a) What is $|0\rangle$ in terms of b particles acting on $|BCS\rangle$?

   (b) What is the variance $(\langle N^2\rangle - \langle N\rangle^2)/\langle N\rangle^2$ in the BCS vacuum for the operator $N = \sum_k a^\dagger_k a_k$? (We used $\langle N\rangle$ in the BCS ground state calculation in class.) Please give your answer in terms of v_k, u_k and $\langle N\rangle$ (if you want to use $\langle N\rangle$).

2. We are going to consider another case, where we have two ground states of *bosonic* operators: $|0\rangle_a$ and $|0\rangle_b$, where

$$a|0\rangle_a = 0 \quad b|0\rangle_b = 0 \tag{5}$$

and

$$\begin{aligned} a &= \alpha b + \beta^* b^\dagger \\ b &= \alpha^* a - \beta^* a^\dagger \end{aligned} \tag{6}$$

with the hermitian conjugates defined in the usual way. Note that generally we could have matrices,

$$\begin{aligned} a_i &= \sum_j \alpha_{ji} b_j + \sum_j \beta^*_{ji} b^\dagger_j \\ b_i &= \sum_i \alpha^*_{ji} a_i - \sum_i \beta^*_{ji} a^\dagger_i \end{aligned} \tag{7}$$

we'll consider the simpler case here.

(a) If we know that

$$[b, b^\dagger] = 1 \tag{8}$$

and want $a, a^\dagger$ to obey bosonic commutation relations as well, what does this require for α, β?

(b) Define $N_b = b^\dagger b$. What is the number density of states of the b particles in the vacuum $|0\rangle_a$?

(c) For the fermionic case, we could find the relation between the two vacua using an arbitrary linear combination of operators, which truncated because no two fermions can have the same quantum numbers. For the bosonic case, this does not happen. We can however still find a vacuum state for one set of operators in terms of the operators and vacuum of another set of operators. Write the state

$$|0\rangle_b = f(a^\dagger)|0\rangle_a \tag{9}$$

as any operators a will annihilate $|0\rangle_a$. Notice that as f is only a function of $a^\dagger$ it can be defined quite easily in terms of a power expansion etc., as any function of $a^\dagger$ commutes with any other function of $a^\dagger$, as long as there is no a involved. We want to solve for $f(a^\dagger)$. Notice that the operator

$$\frac{d}{da^\dagger} \tag{10}$$

obeys the same commutation relations with $a^\dagger$ as a does (in fact, in 221a I believe you showed that $[a, f(a^\dagger)] = \frac{d}{da^\dagger} f(a^\dagger)$).

Write the condition

$$b|0>_b = 0 \tag{11}$$

in terms of a differential equation on the function $f(a^\dagger)$ and use this to write down a solution for $f(a^\dagger)$, up to normalization.

3. Consider the Lagrangian for a particle of mass m and charge q at position $\vec{x}$:

$$L = \frac{1}{2}m\dot{x}^2 - q\phi(x(t),t) + \frac{q}{c}\vec{\dot{x}} \cdot \vec{A}(x(t),t) = T - U \tag{12}$$

where

$$U = q\phi - \frac{q}{c}\vec{\dot{x}} \cdot \vec{A} \tag{13}$$

(a) Find the canonical momentum

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \tag{14}$$

for this system.

(b) What is the Hamiltonian in terms of $\vec{p}, \vec{x}, \vec{A}, \phi$?

(c) Using the Euler-Lagrange equations

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} \tag{15}$$

or the equations of motion using the Hamiltonian, and the definitions of $\vec{E}$ and $\vec{B}$ in terms of the gauge field $\vec{A}(x(t),t)$:

$$\begin{aligned} \vec{E} &= -\vec{\nabla}\phi - \tfrac{1}{c}\tfrac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned} \tag{16}$$

find the equation for

$$\vec{F} = m\vec{\ddot{x}} \tag{17}$$

in terms of $\vec{E}, \vec{B}$.

Note that $\frac{d}{dt}f(x(t),t) \neq \frac{\partial f(x(t),t)}{\partial t}$. Also, in case you've forgotten,

$$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \vec{A} \cdot B_i\vec{C} - (\vec{A} \cdot \vec{B})C_i \tag{18}$$

Please show your work so that the grader can give you credit for effort even if you get the wrong answer. Also, please try to make your work readable!