University of California, Berkeley
Physics 221b Spring 2004

Problem Set 8
due April 9, 2004
corrected

1. In class we quantized the gauge field:

$$\vec{A} = \sum_{k,r} \frac{\vec{\epsilon}_r(k)}{\sqrt{2\omega_k V}}[a_r(k)e^{-i\omega t+i\vec{k}\cdot\vec{x}} + a_r^\dagger(k)e^{i\omega t-\vec{k}\cdot\vec{x}}] \tag{1}$$

and said we could write the electric and magnetic fields in terms of these as

$$\begin{aligned}\vec{E} &= -\tfrac{\partial \vec{A}}{\partial t}\\ \vec{B} &= \vec{\nabla}\times\vec{A}\end{aligned} \tag{2}$$

The Lagrangian

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{3}$$

can be written (taking $A_0 = 0$ as we did)

$$\frac{1}{2}\sum_i(\partial_0 A^i)^2 + (\text{terms without } \partial_0 A_i) \tag{4}$$

and can be used to define a momentum for $\vec{A}$ via

$$P_{A\,i} = \frac{\delta L}{\delta\partial_0 A^i} = -\partial_0 A_i = \partial_0 A^i \tag{5}$$

One way to view the quantization is to impose canonical commutation relations on $\vec{A}$ and its momentum P_A. Using the expansion of A^i above and the definitions in class, calculate the commutator

$$[P_{Ai}(x), A^i(y)]\,. \tag{6}$$

You can leave your answer as a sum over momenta k and polarizations r.

2. Consider a real scalar field (not a vector, like $\vec{A}$) which also can be expanded in oscillators

$$\phi(\vec{x},t) = \sum_{\vec{k}} \sqrt{\frac{1}{2\omega_k V}} (a_k e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + a_k^\dagger e^{i(\omega_k t - \vec{k}\cdot\vec{x})}) \tag{7}$$

where here

$$\omega_k = \sqrt{k^2 + m^2} \tag{8}$$

and

$$[a_k, a_{k'}^\dagger] = \delta_{kk'},\ [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0 \tag{9}$$

Define the average field operator $\bar{\phi}$ in the vicinity of the origin by

$$\bar{\phi} = \frac{1}{(2\pi b^2)^{3/2}} \int d^3x e^{-\frac{r^2}{2b^2}} \phi(\vec{x},t)\ . \tag{10}$$

Take $b \ll m^{-1}$ and find the dependence of $\langle 0|\bar{\phi}^2|0\rangle$ on b to leading order (i.e. zeroth) in $mb \equiv a$.

3. Consider a coherent state $|\alpha\rangle$,

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger} |0\rangle\ . \tag{11}$$

Take $a^\dagger$ to be the creation operator for a photon of energy $\omega_{\tilde{k}} \equiv \omega$ and momentum $\vec{\tilde{k}}$and a particular polarization $\tilde{r}$.

   (a) What are the fluctuations in number $a^\dagger a$ in the state $|\alpha\rangle$? (That is, $\langle N^2\rangle - \langle N\rangle^2$?)

   (b) Writing the gauge field

$$\vec{A} = \sum_{k,r} \frac{\vec{\epsilon}_r(k)}{\sqrt{2\omega_k V}} [a_r(k) e^{-i\omega t + i\vec{k}\cdot\vec{x}} + a_r^\dagger(k) e^{i\omega t - \vec{k}\cdot\vec{x}}] = \sum_{k,r} \vec{A}_{k,r} \tag{12}$$

   as above, what is

$$\langle\alpha|\vec{E}|\alpha\rangle\ ? \tag{13}$$

   Write this in terms of the magnitude and phase of $\alpha = |\alpha| e^{i\phi}$.

(c) Define

$$E_{\mathbf{k},\mathbf{r}} = -\partial_0 \vec{A}_{k,r} \tag{14}$$

What is

$$(\Delta E_{k,r})^2 = \langle\alpha|\vec{E}_{k,r}^2|\alpha\rangle - (\langle\alpha|\vec{E}_{k,r}|\alpha\rangle)^2 \ ? \tag{15}$$

(d) Summing over the energy fluctuation diverges, but in class we took an average over a box the size of the wavelength of the field in question when considering a box with only one wavelength of radiation in it. We argued that the energy fluctuations would be relatively small for an n-photon state if there were a lot of photons, i.e. if

$$\frac{n}{V} \gg \frac{1}{\lambda^3} \ . \tag{16}$$

In what limit do the fluctuations in the number of photons relative to the average become small for the coherent state above? Is this compatible with the "classical regime" for the energy in the corresponding n photon state?

Please show your work so that the grader can give you credit for effort even if you get the wrong answer. Also, please try to make your work readable!