University of California, Berkeley
Physics 221b Spring 2004

Problem Set 9
due April 16, 2004

1. (Sakurai 2-5) The phenomenological interaction Hamiltonian responsible for the decay of the Σ^0 hyperon ($\Sigma^0 \to \Lambda + \gamma$) located at $\vec{x} = 0$ can be taken as

$$\frac{\kappa e \hbar}{(m_\Lambda + m_\Sigma)c} \tau_{\Lambda\Sigma} \vec{\sigma} \cdot (\vec{\nabla} \times \vec{A})|_{x=0} \tag{1}$$

where $\tau_{\Lambda\Sigma}$ is an operator that converts Σ^0 into Λ, leaving the spin state unchanged, and κ is a dimensionless constant assumed to be of the order of unity. Note that Σ^0 is a particle with spin 1/2, made of a u, d, s quark combination.

   (a) By calculating the transition rate as a function of solid angle, $w d\Omega$, show that the angular distribution of the decay is isotropic even when the parent Σ^0 is polarized. (It is useful to first calculate the decay rate into a particular spin state of $\Lambda = \Lambda_s$ polarization with a photon r in terms of the matrix element of the operator above.)

   (b) Assuming this is practically the only decay channel, find the mean lifetime (in seconds) for $\kappa = 1$ ($m_\Lambda = 1115\text{Mev}/c^2$, $m_\Sigma = 1192$ MeV$/c^2$). (Note that κ is not necessarily 1, just calculate this case.)

   The point of this part of the problem is to get some experience putting back in the $\hbar$ and c constants to get dimensionful things. It's useful to know that $\alpha = e^2/(4\pi) = 1/137$ in the units we've been using and where $c = \hbar = 1$ and that $\hbar = 6.58 \times 10^{-22}$MeV sec when you want to put it back in.

2. Calculate the spontaneous emission rate (dipole transition) of a hydrogen (like) atom going from energy level $n = 2, \ell = 1$ to $n = 1, \ell = 0$. You can assume that you have the rate already, i.e.

$$w = \frac{e^2 \omega^3}{3\pi} |\langle A|\vec{x}|B\rangle|^2 \tag{2}$$

where A and B are the final and inital atomic states. You may find the following formulae useful

$$\begin{array}{l} \psi_{100} = (\frac{1}{\pi a_0^3})^{1/2} e^{-r/a_0} \\ \psi_{210} = (\frac{1}{32\pi a_0^3})^{1/2} \frac{r}{a_0} \cos\theta e^{-r/2a_0} \\ \psi_{21\pm1} = (\frac{1}{64\pi a_0^3})^{1/2} \frac{r}{a_0} \sin\theta e^{\pm i\phi} e^{-r/2a_0} \\ a_0 = 0.53 \times 10^{-8} cm \\ E_n = -\frac{\alpha^2}{2a_0 n^2} \end{array} \tag{3}$$

Note that you can choose the initial atom to be in any 2p state you want, so pick a convenient one!

3. In class I outlined the calculation of the Thompson cross section for the scattering of photons of polarization r and wavevector $\vec{k}$ into photons of polarization r' and wavevector $\vec{k}'$, with the photon energy large compared to the binding energy of the atom (so electron is essentially free) and small compared to the rest mass of the electron (so we can use non-relativistic theory and neglect recoil), i.e. $k^2 = \omega^2 \sim k'^2$. The rate for scattering into polarization r from polarization r' is given by

$$w_r d\Omega = 2\pi |\langle f|H_{int}|i\rangle|^2 \frac{V}{(2\pi)^3}\omega^2 \tag{4}$$

and we consider the part of H_{int} quadratic in $\vec{A}$, i.e.

$$\frac{e^2}{2m}\vec{A}\cdot\vec{A} \tag{5}$$

where

$$\vec{A} = \sum_{r,k} \frac{\vec{\epsilon}_r(\vec{k})}{\sqrt{2\omega_k V}} [a_r(\vec{k}) e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + a_r^\dagger(\vec{k}) e^{+i(\omega_k t - \vec{k}\cdot\vec{x})}] \tag{6}$$

(a) Taking the interaction to happen at $\vec{x} = 0$, calculate the matrix element between two one-photon states:

$$\langle f|H_{int}|i\rangle = \langle \vec{k}', r'|\vec{A}\cdot\vec{A}|\vec{k}, r\rangle \tag{7}$$

where $|\vec{k}, r\rangle = |n_r(\vec{k}) = 1\rangle$, etc. Only consider terms which annihilate the incoming photon and create an outgoing photon, and take $k^2 \sim k'^2 \sim \omega^2$. There are also terms which create a second photon (in addition to the first one) and then annihilate it, the "self-energy" of the electron, which describes a different process, which is not of interest here, but would contribute if you took $\vec{k} = \vec{k'}$ and $r = r'$. (For more on this you can read Sakurai pp. 64ff.) You will find that the matrix element goes as $\frac{1}{V}$ as we saw in class, due to the presence of two fields $\vec{A}$.

(b) Squaring the matrix element $\frac{e^2}{2m}\langle \vec{k'}, r'|\vec{A} \cdot \vec{A}|\vec{k}, r\rangle$, combining with the phase space factors above, and dividing by the incident flux of photons (velocity/volume $= c/V = 1/V$), we find a cross section

$$\sigma_{r,\vec{k}\to r',\vec{k'}} d\Omega = (\frac{e^2}{4\pi m})^2 (\vec{\epsilon}_r(\vec{k}) \cdot \vec{\epsilon}_{r'}(\vec{k'}))^2 d\Omega = r_0^2 (\vec{\epsilon}_r(\vec{k}) \cdot \vec{\epsilon}_{r'}(\vec{k'}))^2 d\Omega \tag{8}$$

where r_0 is the classical radius of the electron $\sim 10^{-15}$ m.

To consider the total cross section for unpolarized light one can i.e. sum over initial polarizations and then divide by 2, and sum over final polarizations. So we require

$$\frac{1}{2}\sum_r \sum_{r'} (\vec{\epsilon}_r(\vec{k}) \cdot \vec{\epsilon}_{r'}(\vec{k'}))^2 \tag{9}$$

Express this in terms of the wavevectors of the final and initial photons, $\hat{k}$ and $\hat{k}'$. Remember from class that for any vector

$$\vec{v} = (\vec{v} \cdot \vec{\epsilon}_1)\vec{\epsilon}_1 + (\vec{v} \cdot \vec{\epsilon}_2)\vec{\epsilon}_2 + (\vec{v} \cdot \hat{k})\hat{k} \tag{10}$$

and the consequences of this for $\vec{v} \cdot \vec{v}$. So one way to do this is to write $\vec{\epsilon}_{r'}(\vec{k'})$ in terms of its projection on $\vec{\epsilon}_r(\vec{k})$ and $\hat{k}$ and separate out the sum over r, and then to project $\hat{k}$ on $\vec{\epsilon}_{r'}(\vec{k'})$ etc. and do the second sum.

You can check your result by seeing if the integrated cross section

$$\int d\Omega \sigma = \frac{8\pi}{3} r_0^2 \tag{11}$$

Please show your work so that the grader can give you credit for effort even if you get the wrong answer. Also, please try to make your work readable!