Physics 7C, Spring 2003

Notes on sums for interference and diffraction

We considered what sort of interference pattern we get when we have a slit of finite width a and/or a set of slits all of finite width. The basic idea is that you add up all the phase shifts for all of them, and that generally

$$E_{tot} = \sum E_i \tag{1}$$

where E_i is the wave from each slit (or part of slit) and that

$$I_{tot} \propto |\sum E_i|^2 \tag{2}$$

As you first sum and then square, the various waves can cancel out parts of each other. This is the essence of interference.

We generally took a set of slits in a vertical direction, and measured the interference pattern at an angle θ from the horizontal. We assumed we were doing this very far away so that the difference in angle between rays going from the two slit positions and ending up at the same point on the screen was negligible.

For two slits separated by distance d in the vertical direction, rays leaving at angle θ will travel a distance $d \sin\theta$ more for the bottom slit compared to the top slit (measuring θ as increasing counterclockwise). This means a change in phase $\delta = kd\sin\theta$ where $k = 2\pi/\lambda$.

Note that in the below the wave is always taken to be $E_0 \cos(kx - \omega t)$ when it has zero phase. That is, we said that whatever direction it was travelling was the x direction. When we add up waves that aren't going perfectly horizontally, then the direction of the wave isn't in the horizontal direction, so we should think of x as being in the direction of travel of the wave, not in the horizontal direction. Often in books people write this direction instead as r.

Double slit

For the simple double slit, we take two waves separated by distance d in the vertical direction and look at the pattern at angle θ from the horizontal. As we just said, they differ in path by $kd\sin\theta$. So we have

$$\begin{aligned}
&E_{tot} = E_1 \cos(kx - \omega t) + E_2 \cos(kx - \omega t + \delta) \\
&|E_{tot}|^2 = E_1^2 \cos^2(kx - \omega t) + E_2^2 \cos^2(kx - \omega t + \delta) + 2E_1E_2 \cos(kx - \omega t)\cos(kx - \omega t + \delta) \\
&= E_1^2 \cos^2(kx - \omega t) + E_2^2 \cos^2(kx - \omega t + \delta) + E_1E_2 \cos(2kx - 2\omega t + \delta) + E_1E_2 \cos(\delta)
\end{aligned} \tag{3}$$

So, for the time average,

$$< |E_{tot}|^2 >_{time\ average} = \frac{1}{2}E_1^2 + \frac{1}{2}E_2^2 + E_1 E_2 \cos\delta \tag{4}$$

where we've used that

$$\begin{array}{l} < \cos^2(kx - \omega t) >= \frac{1}{2} \\ < \cos(kx - \omega t) >= 0 \end{array} \tag{5}$$

in the time averages. So for the double slit we have $E_1 = E_2$ (same source usually, a hole right before both slits) and $\delta = kd\sin\theta$ so we get

$$\begin{array}{rl} < |E_{tot}|^2 >_{time\ average} & = \frac{1}{2}E_1^2 + \frac{1}{2}E_1^2 + E_1 E_1 \cos\delta \\ & = E_1^2(1 + \cos(kd\sin\theta)) \\ & = 2E_1^2\cos^2(\frac{kd}{2}\sin\theta) \end{array} \tag{6}$$

Diffraction by a single slit

For a single slit of finite width a, we just add up all the phases from every position on the slit. If the slit is of height a, take the relative phases from the center of the slit, so that the height changes from $a/2$ to $-a/2$ going down the slit. To get to a point far away at an angle θ from the horizontal, relative to the height zero middle of the slit , a ray from the top of the slit travels a distance $a/2\sin\theta$ less, and thus has a phase of $ka/2\sin\theta$ smaller. A ray from the bottom, relative to the center, travels a distance $a/2\sin\theta$ more and thus has a phase $ka/2\sin\theta$ larger relative to the ray from the center. Generally, if the height is y, $-a/2 \leq y \leq a/2$, then the phase of a ray coming from position y on the slit hitting a screen at angle θ is $-ky\sin\theta$ relative to the ray from the center. ($y = 0$ at the center.)[1]

[1] There is a subtlety involved in integrating over the y direction, as Edy doesn't have the dimensions of E, as y is a length. So it is necessary to divide by a scale to get something that scales as E. In one lecture I wrote this scale as ℓ as it doesn't show up in the result, because we generally are considering ratios of intensities. I set the scale equal to the wavelength over 2π, i.e. we multiplied by k, in the next lecture. Another convention is to consider E_0 as the total field going through the slit, so that the part going through the slit in the region of size dy/a is $E_0 dy/a$. One can choose a convention for this because it is only part of the intensity that is being considered here. The field is squared and then related to the intensity via some constant, which will be different for different conventions. The general formula for the intensity, which we haven't and won't cover in detail, depends both upon the wavelength and the distance to the screen R, as $k/R = 2\pi/(\lambda R)$. It depends on two distances, because when integrating over a slit generally the slit will have width in two directions. Below is the calculation with the factor of k in front.

To see what happens, we add these up:

$$\begin{aligned}
E_{tot} &= \int_{-a/2}^{a/2} kE_0 \cos(kx - \omega t + \delta(y))dy \\
&= \int_{-a/2}^{a/2} kE_0 \cos(kx - \omega t - ky\sin\theta) \\
&= -E_0 k \frac{\sin(kx-\omega t-ky\sin\theta)}{k\sin\theta}\Big|_{y=-a/2}^{y=a/2} \\
&= -\frac{E_0 k}{k\sin\theta}[\sin(kx - \omega t - ka/2\sin\theta) - \sin(kx - \omega t + ka/2\sin\theta)] \\
&= 2E_0 ka \frac{\cos(kx-\omega t)\sin(\frac{ka}{2}\sin\theta)}{ka\sin\theta} \\
&= E_1(\theta = 0)\frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta}
\end{aligned} \tag{7}$$

On the last line, the definition

$$\begin{aligned}
E_1(\theta = 0) &= \int_{-a/2}^{a/2} kE_0 \cos(kx - \omega t + \delta(y)_{\theta=0})dy \\
&= E_0 k\cos(kx - \omega t)\int_{=a/2}^{a/2} dy \\
&= E_0 ka\cos(kx - \omega t)
\end{aligned} \tag{8}$$

was used, that is, at $\theta = 0$ all the phases $\delta(y)$ are zero. (The two in front of E_0 was absorbed in the denominator to give $ka/2$.) So we can relate the field at any angle to the field at $\theta = 0$.

The intensity is proportional to $|E|^2$:

$$I = I_1(\theta = 0)|\frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta}|^2 \tag{9}$$

where

$$I_1(\theta = 0) \propto |E_0 ka\cos(kx - \omega t)|^2 \tag{10}$$

which time averages to

$$< I_1(\theta = 0) > \propto \frac{1}{2}|E_0 ka|^2 \tag{11}$$

so that the general time averaged intensity for the single slit is

$$< I >=< I_1(\theta = 0) > |\frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta}|^2 \tag{12}$$

This has minima when the numerator is zero, ie when $\frac{ka}{2}\sin\theta = m\pi$ or in terms of wavelength $\lambda = 2\pi/k$,

$$a\sin\theta = m\lambda\ . \tag{13}$$

The maxima are a bit harder to find, they are extrema of the expression with respect to e.g. $\sin\theta$. There is a central maximum at $\theta = 0$ because $\frac{\sin x}{x} \to 1$ when $x \to 0$.

Two slit plus diffraction

Again, we want to sum up all the rays that go to a point at angle θ above the horizontal on the screen. For each slit we get an integral over the slit and then we sum over the slits. Taking the slits to all be the same width a and defining the phases relative to the center of each slit, we can just combine the above calculations.

$$E = \sum_{slits} \int_{1slit} E_0 k \cos(kx - \omega t + \delta(y) + \delta_{slit}) dy \tag{14}$$

In words, we want to sum up every ray coming through. so we do the sum in two parts. First we sum up the ray for each slit, this gives us, for each slit

$$E_0 ka \cos(kx - \omega t + \delta_{slit}) \frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta} \tag{15}$$

and then we sum up the different slits, using for δ_{slit} the relative phases for the different slits. E.g. for the two slit case, the phase difference between the top and bottom wave is $kd\sin\theta$ if the slits are a distance d apart and we are looking at angle θ. For convenience, we take the top to have phase $-\frac{kd}{2}\sin\theta$ and the bottom to have phase $\frac{kd}{2}\sin\theta$. If we chose some other phases, but with the same phase difference, we would get the same result at the end, when we calculated the intensity, but this gives the form the most directly. So we get

$$\begin{aligned} E_{tot} &= E_{top} + E_{bot} \\ &= (E_0 ka \cos(kx - \omega t + \delta_{slit,top}) + E_0 ka \cos(kx - \omega t + \delta_{slit,bottom})) \frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta} \\ &= (E_0 ka \cos(kx - \omega t - \frac{kd}{2}\sin\theta) + E_0 ka \cos(kx - \omega t + \frac{kd}{2}\sin\theta)) \frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta} \\ &= 2E_0 ka \cos(kx - \omega t) \cos(\frac{kd}{2}\sin\theta) \frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta} \end{aligned} \tag{16}$$

This turns into an time averaged intensity

$$\begin{aligned} < I > &\propto < |E|^2 > \\ &= 2E_0^2 (ka)^2 |\cos(\frac{kd}{2}\sin\theta)|^2 |\frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta}|^2 \end{aligned} \tag{17}$$

or in terms of the single slit,

$$I = 4 < I_1(\theta=0) > |\cos(\frac{kd}{2}\sin\theta)|^2 |\frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta}|^2 \qquad (18)$$

The first factor is the interference pattern, and it is multiplied by the diffraction pattern. As the spacing between the slits is usually bigger than the slit width, generally you will have a lot of interference maxima inside of the main central diffraction peak. There is a picture of what the two different terms look like and their product in Giancoli's book.

Multiple slits: diffraction grating

We will look at a bunch of slits, all of the same width a. We again will use

$$E = \sum_{slits} \int_{1slit} E_0 k \cos(kx - \omega t + \delta(y) + \delta_{slit}) dy \qquad (19)$$

and since the slits are all of the same width a, we get the same prefactor in front of each and thus get

$$E = \sum_{slits} E_0 ka \cos(kx - \omega t + \delta_{slit}) \frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta} \qquad (20)$$

To do the sum over the slits, we need more details about them. Take the following case:

Consider N slits, N large, all separated from each other by distance d. We will take N odd and choose the phase at the center slit to be zero. So we have $(N-1)/2$ slits above this slit and $(N-1)/2$ slits below the center slit. We can label the slit as the nth slit, with the top one being $n = (N-1)/2$, the middle one being $n = 0$ and the bottom one being $n = -(N-1)/2$. The calculation will work for other labellings of course but this one makes it the simplest. For other labellings the algebra is harder and you of course always end up with the same relative phases between rays but the overall phase, which falls out, is different. Consider a bunch of rays going through these slits and meeting on a screen, again very far away. Take the angel of these rays with the horizontal to be θ, as usual. Relative to the center slit, a ray going out of the top slit will travel a path that is $\frac{(N-1)}{2} d\sin\theta$ shorter, and a ray going out of the bottom slit will travel a path that is $\frac{(N-1)}{2} d\sin\theta$ longer. In general, the nth ray, using the definition of n above, will travel a distance $nd\sin\theta$ relative to the center ray. So the phase for the nth ray will be $k \times distance = knd\sin\theta$ relative to the ray in the center. The sum over all the different fields with their phases is thus

$$E = \sum_{n=-(N-1)/2}^{(N-1)/2} E_0 ka \cos(kx - \omega t - knd\sin\theta) \frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta} \tag{21}$$

You can do this using cosine identities over and over again, but there is a shorter way. Writing

$$e^{ix} = \cos(x) + i\sin(x) \tag{22}$$

we can say that

$$\cos(kx - \omega t - knd\sin\theta) = Real[e^{i(kx-\omega t - knd\sin\theta)}] \tag{23}$$

and so that

$$\sum_{n=-(N-1)/2}^{(N-1)/2} \cos(kx - \omega t - knd\sin\theta) = \sum_{n=-(N-1)/2}^{(N-1)/2} Real[e^{i(kx-\omega t - knd\sin\theta)}] \tag{24}$$

as the real part of the sum of two numbers is equal to the sum of the real parts of both numbers, we can write this as

$$\begin{aligned} \textstyle\sum_{n=-(N-1)/2}^{(N-1)/2} \cos(kx - \omega t - knd\sin\theta) &= Real[\textstyle\sum_{n=-(N-1)/2}^{(N-1)/2} e^{i(kx-\omega t - knd\sin\theta)}] \\ &= Real[e^{i(kx-\omega t)} \textstyle\sum_{n=-(N-1)/2}^{(N-1)/2} e^{-iknd\sin\theta}] \end{aligned} \tag{25}$$

Lets just look at the sum part of this:

$$\sum_{n=-(N-1)/2}^{(N-1)/2} e^{-iknd\sin\theta} = \sum_{n=-(N-1)/2}^{(N-1)/2} \tilde{a}^n \tag{26}$$

where

$$\tilde{a} = e^{-ikd\sin\theta} \tag{27}$$

This is a geometric series, and one can solve it by noting that

$$\begin{aligned} \textstyle\sum_{n=-(N-1)/2}^{(N-1)/2} \tilde{a}^n &= \tilde{a}^{-(N-1)/2} + \tilde{a}^{-(N-1)/2+1} + \cdots + \tilde{a}^{(N-1)/2} \\ \tilde{a}\textstyle\sum_{n=-(N-1)/2}^{(N-1)/2} \tilde{a}^n &= \tilde{a}^{-(N-1)/2+1} + \tilde{a}^{-(N-1)/2+1} + \cdots + \tilde{a}^{(N-1)/2+1} \end{aligned} \tag{28}$$

The middle terms are the same for both, only the ends are different. So if you subtract, you get

$$(1-\tilde{a}) \sum_{n=-(N-1)/2}^{(N-1)/2} \tilde{a}^n = \tilde{a}^{-(N-1)/2} - \tilde{a}^{(N-1)/2+1} \tag{29}$$

or

$$\begin{aligned} \textstyle\sum_{n=-(N-1)/2}^{(N-1)/2} \tilde{a}^n &= \frac{\tilde{a}^{-(N-1)/2} - \tilde{a}^{(N-1)/2+1}}{1-\tilde{a}} \\ &= \frac{\tilde{a}^{-(N-1)/2-1/2} - \tilde{a}^{(N-1)/2+1-1/2}}{\tilde{a}^{-1/2} - \tilde{a}^{1/2}} \end{aligned} \tag{30}$$

Plugging in for $\tilde{a} = e^{-ikd\sin\theta}$ we get

$$\sum_{n=-(N-1)/2}^{(N-1)/2} e^{-iknd\sin\theta} = \frac{e^{i\frac{N}{2}kd\sin\theta} - e^{-i\frac{N}{2}kd\sin\theta}}{e^{i\frac{kd}{2}\sin\theta} - e^{-i\frac{kd}{2}\sin\theta}} = \frac{\sin(\frac{N}{2}kd\sin\theta)}{\sin(\frac{kd}{2}\sin\theta)} \tag{31}$$

This is real, so plugging in to the above starting expression

$$\begin{aligned} &Real[e^{i(kx-\omega t)} \textstyle\sum_{n=-(N-1)/2}^{(N-1)/2} e^{-iknd\sin\theta}] = Real[e^{i(kx-\omega t)}]\frac{\sin(\frac{N}{2}kd\sin\theta)}{\sin(\frac{kd}{2}\sin\theta)} \\ &= \cos(kx-\omega t)\frac{\sin(\frac{N}{2}kd\sin\theta)}{\sin(\frac{kd}{2}\sin\theta)} \end{aligned} \tag{32}$$

So going back to the starting expression for the total E field

$$\begin{aligned} E &= \textstyle\sum_{n=-(N-1)/2}^{(N-1)/2} E_0 ka\cos(kx-\omega t - knd\sin\theta)\frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta} \\ &= E_0 ka\cos(kx-\omega t)\frac{\sin(\frac{N}{2}kd\sin\theta)}{\sin(\frac{kd}{2}\sin\theta)}\frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta} \\ &= E_1(\theta=0)\frac{\sin(\frac{N}{2}kd\sin\theta)}{\sin(\frac{kd}{2}\sin\theta)}\frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta} \end{aligned} \tag{33}$$

with resulting intensity

$$I = I_1(\theta=0)\left|\frac{\sin(\frac{N}{2}kd\sin\theta)}{\sin(\frac{kd}{2}\sin\theta)}\right|^2\left|\frac{\sin(\frac{ka}{2}\sin\theta)}{\frac{ka}{2}\sin\theta}\right|^2 \tag{34}$$

Note what happens when $N=2$, and you get the double slit again!

So again, we get the interference pattern times the diffraction pattern. For this interference pattern, the maxima are when the denominator goes to zero, i.e.

$$\frac{kd}{2}\sin\theta = m\pi \tag{35}$$

which can be rewritten

$$d \sin\theta = m\lambda \tag{36}$$

That is, the path difference between the first and second slit and first and thousandth slit etc etc are all integer multiples of the wavelength. So you get constructive interference in this case. The sum doesn't diverge, because the interference factor is

$$\left|\frac{\sin(\frac{N}{2}kd\sin\theta)}{\sin(\frac{kd}{2}\sin\theta)}\right|^2 \tag{37}$$

which has both the numerator and denominator going to zero at the same time. In fact $\frac{\sin(Nx)}{\sin(x)} \sim N\frac{\cos(Nx)}{\cos(x)}$ when $x \to m\pi$, which is $N(-1)^{(m(N-1))}$. That is, the intensity will go as N^2 where N is the number of slits. [2]

Note also that if we say

$$d\sin\theta = m\lambda \to \sin\theta = m\frac{\lambda}{d} \tag{39}$$

and so for some m, this would imply that $\sin\theta > 1$, which isn't possible. so all values of m do not appear. Generally what happens is that the maxima for large N are bigger than for small N and more well separated. So this can be used to sort out a wave into different wavelengths, and is better than a prism as it doesn't absorb the electromagnetic wave.

[2] There is a limit, where the number of slits gets very big and the spacing very small, where this starts looking like the diffraction pattern again: if all the slits are within a region of size a, so that N gets very large as d gets small, and $Nd \to a$, then we get for the interference factor

$$\frac{\sin(\frac{Nd}{2}k\sin\theta)}{\sin(\frac{kd}{2}\sin\theta)} \to_{N\ big,\ dsmall,\ Nd=a\ fixed} \frac{\sin(\frac{a}{2}k\sin\theta)}{\frac{kd}{2}\sin\theta} = N\frac{\sin(\frac{a}{2}k\sin\theta)}{\frac{ka}{2}\sin\theta} \tag{38}$$

because for small enough d, $\sin\frac{kd}{2}\sin\theta \sim \frac{kd}{2}\sin\theta$, while the numerator is larger by a factor of N, which is big, and thus can't necessarily be approximated the same way. In this limit we see that the many slit interference looks like a single slit diffraction pattern.