lecture notes for Jan. 24, 2003, Physics 7c

1. Last Time

Last time, we defined

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{1}$$

and rewrote Maxwell's equations, in the case of no sources ($Q = \rho = \vec{J} = 0$) as

$$\begin{aligned} &\vec{\nabla}\cdot\vec{E} = 0 \\ &\vec{\nabla}\cdot\vec{B} = 0 \\ &\vec{\nabla}\times\vec{E} = -\tfrac{\partial\vec{B}}{\partial t} \\ &\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\tfrac{\partial\vec{E}}{\partial t} \end{aligned} \tag{2}$$

We saw in the applet (at http://www.cco.caltech.edu/~phys1/java/phys1/MovingCharge/MovingCharge.html) how these fields, away from a source, could be set up by an accelerating charge. (The applet is linked off of Key Ideas on the Web Page).

2. Wave Equation

First we are going to show how Maxwell's equations, above, give the wave equation. For this, we will take the derivative of the equations for $\vec{\nabla}\times\vec{E}$ and $\vec{\nabla}\times\vec{B}$. That is, we will say that since

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{3}$$

from above, that

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t}\ . \tag{4}$$

Now we can rewrite both sides:

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\nabla^2\vec{E} + \nabla(\nabla\cdot\vec{E}) \tag{5}$$

which you can see by writing out in gory detail (true for any vector, actually), and

$$-\vec{\nabla}\times\frac{\partial\vec{B}}{\partial t} = -\frac{\partial(\vec{\nabla}\times\vec{B})}{\partial t}\ . \tag{6}$$

Putting this together we get:

$$-\nabla^2\vec{E} + \nabla(\nabla\cdot\vec{E}) = -\frac{\partial(\vec{\nabla}\times\vec{B})}{\partial t} \tag{7}$$

which is very convenient, because we recognize the terms in parenthesis, and can eliminate them using Maxwell's other equations to get

$$-\nabla^2\vec{E} = -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}\ . \tag{8}$$

or

$$\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}\ . \tag{9}$$

We can also get the same thing for the magnetic field $\vec{B}$. We start with

$$\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \tag{10}$$

and act with $\vec{\nabla}\times$ on both sides,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \mu_0\epsilon_0\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t}\ . \tag{11}$$

and use the same identities as above,

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = -\nabla^2\vec{B} + \nabla(\nabla\cdot\vec{B}) \tag{12}$$

and

$$\mu_0\epsilon_0\vec{\nabla}\times\frac{\partial\vec{E}}{\partial t} = \mu_0\epsilon_0\frac{\partial(\vec{\nabla}\times\vec{E})}{\partial t}\ . \tag{13}$$

and then using $\vec{\nabla}\cdot\vec{B} = 0$ and $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ and repeating as we did above, we get for $\vec{B}$ that

$$\nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}\ . \tag{14}$$

As

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{15}$$

we have for each component of $\vec{E},\vec{B}$ that

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})E_x = \mu_0\epsilon_0\frac{\partial\vec{E}_x}{\partial t^2} \tag{16}$$

etc, and same for E_y, E_z, B_x, B_y, B_z.

That is, all the components of the electromagnetic field satisfy the wave equation!!

In the wave equation we generally have

$$\nabla^2\vec{E} = \frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t^2}\ . \tag{17}$$

that is, the velocity of the wave in our case obeys

$$v^2 = \frac{1}{\mu_0 \epsilon_0} \tag{18}$$

and when you plug in numbers you get

$$v = 3.0 \times 10^8 \mathrm{m\ s}^{-1} \tag{19}$$

This is using ϵ_0 and μ_0 so this is the speed of light in vacuum so to speak. Generally this is denoted by c as it is a constant of nature.

An example of a solution to the wave equation is the function

$$A(x,t) = A_{right} \cos(kx - \omega t) \tag{20}$$

which describes a wave travelling to the right along the x axis. The distance between crests is the wavelength, $\lambda = 2\pi/k$, and the amount of time between the wave hitting a crest twice in a row at the same point is the period, T, where $T = 2\pi/\omega = 1/f$. Both ω and f can be called frequency (sometimes people call ω angular frequency to avoid confusion). Because of the wave equation, we know that $\lambda f = c$ or in terms of k, ω, $k = \omega/c$. The size of the wave (amplitude) is given by $|A|$. More generally the wave can travel not along the x axis but along any axis, and have an arbitrary constant phase also, and also go both left and right: i.e.

$$A_{right} \cos(\vec{k} \cdot \vec{x} - \omega t + \phi_r) + A_{left} \cos(\vec{k} \cdot \vec{x} + \omega t + \phi_l) \tag{21}$$

However, all of the equations we are using are linear, i.e. if we have one solution $A_1(\vec{x},t)$ to the wave equation, and another solution $A_2(\vec{x},t)$ to the wave equation, we can add them to get a third solution which also works, $A_1(\vec{x},t) + A_2(\vec{x},t)$. So we will use this as a reason to look for the simplest solution to the wave equation we can find, to get a more general configuration we can always add in other simple solutions as well. In a specific case we will want a configuration that satisfies whatever conditions we've got in that particular case (e.g. initial conditions or boundary conditions).

3. Electromagnetic Waves

So we know now that solutions to Maxwell's equations in vacuum are waves travelling at the speed of light. That is $\vec{E}, \vec{B}$ both have components that oscillate as solutions to the wave equation.

However, not any solution to the wave equation will also satisfy Maxwell's equations, there is more information. Let's look for a solution of the wave equation that satisfies Maxwell's equations too.

We will start with a simple trial solution and then see what it has to satisfy in order to solve Maxwell's equations. We will want it to be a wave travelling to the right along the x-axis, and so it will have time dependence of the form $\cos(kx - \omega t + \phi)$. We will take the phase ϕ to be zero and see that it works. If $\phi = -\pi/2$ then we would be looking at $\sin(kx - \omega t)$.

So we will start with

$$\begin{aligned}
\vec{E}_{test} &= \hat{x}E_x + \hat{y}E_y + \hat{z}E_z \\
&= \hat{x}E_{xc}\cos(kx - \omega t) + \hat{y}E_{yc}\cos(kx - \omega t) + \hat{z}E_{zc}\cos(kx - \omega t) \\
\vec{B}_{test} &= \hat{x}B_x + \hat{y}B_y + \hat{z}B_z \\
&= \hat{x}B_{xc}\cos(kx - \omega t) + \hat{y}B_{yc}\cos(kx - \omega t) + \hat{z}B_{zc}\cos(kx - \omega t)
\end{aligned} \tag{22}$$

We have assumed that the phase is the same for all of them (all cosines) and will see that this solves Maxwell's equations. Making them all sines will work too, and in general, using superposition, you could add both together if you wanted.

So this is our test solution. We want to solve for $E_x, E_y, E_z, B_x, B_y, B_z$, all constants. For a general solution to the wave equation they could be anything, but for an electromagnetic wave, there are some constraints.

First we consider the first two Maxwell equations:

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} = 0 \\
\vec{\nabla} \cdot \vec{B} = 0
\end{aligned} \tag{23}$$

In terms of our equations, this is

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial(E_{xc}\cos(kx - \omega t))}{\partial x} + \hat{y}\frac{\partial(E_{yc}\cos(kx - \omega t))}{\partial y} + \hat{z}\frac{\partial(E_{zc}\cos(kx - \omega t))}{\partial z} = 0 \tag{24}$$

Since we only have x dependence though, we find that

$$\vec{\nabla} \cdot \vec{E} = \frac{\partial(E_{xc}\cos(kx - \omega t))}{\partial x} = -kE_{xc}\sin(kx - \omega t) = 0\ . \tag{25}$$

The function $\sin(kx - \omega t)$ can't be zero unless both $k, \omega = 0$ and if $k = 0$ then we have a very sick wave (wavelength infinite). The other choice is that our constant coefficient E_{xc} must be zero in order to have a solution.

The same thing happens with B_{xc} when you plug in. So now we see that in order to be a solution, we have constrained both E_{xc} and B_{xc}.

Note: This means in particular that the wave is oscillating perpendicular to its direction of motion, it is *transverse*. This is a general property of solutions to Maxwell's equations in vacuum, not just the fact that we took a simple solution to look at.

So now we are left with a trial solution:

$$\begin{aligned}
\vec{E}_{test} &= \hat{x}E_x + \hat{y}E_y + \hat{z}E_z \\
&= \hat{y}E_{yc}\cos(kx - \omega t) + \hat{z}E_{zc}\cos(kx - \omega t) \\
\vec{B}_{test} &= \hat{x}B_x + \hat{y}B_y + \hat{z}B_z \\
&= \hat{y}B_{yc}\cos(kx - \omega t) + \hat{z}B_{zc}\cos(kx - \omega t)
\end{aligned} \tag{26}$$

We still have two more equations to impose, and they will relate the E and B constants. Let's look at the equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{27}$$

and plug in:

$$\vec{\nabla} \times \vec{E} = \hat{x}(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial x}) + \hat{y}(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}) + \hat{z}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) \tag{28}$$

Now, this is simpler than it looks, because we only have x dependence! So for our case

$$\begin{aligned} \vec{\nabla} \times \vec{E}_{test} &= -\hat{y}\tfrac{\partial E_z}{\partial x} + \hat{z}\tfrac{\partial E_y}{\partial x} \\ &= \hat{y}kE_{zc}\sin(kx - \omega t) - \hat{z}kE_{yc}\sin(kx - \omega t) \quad \textit{our case} \end{aligned} \tag{29}$$

Also,

$$-\frac{\partial \vec{B}}{\partial t} = -\hat{y}\omega B_{yc}\sin(kx - \omega t) - \hat{z}\omega B_{zc}\sin(kx - \omega t) \; \textit{our case} \tag{30}$$

So putting this together we get

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= -\tfrac{\partial \vec{B}}{\partial t} \\ \hat{y}kE_{zc}\sin(kx - \omega t) - \hat{z}kE_{yc}\sin(kx - \omega t) &= -\hat{y}\omega B_{yc}\sin(kx - \omega t) - \hat{z}\omega B_{zc}\sin(kx - \omega t) \end{aligned} \tag{31}$$

As this is true for all x, t, we can divide by $\sin(kx - \omega t)$ to get

$$\hat{y}kE_{zc} - \hat{z}kE_{yc} = -\hat{y}\omega B_{yc} - \hat{z}\omega B_{zc} \tag{32}$$

So we see that the constants we get are related. The directions $\hat{y}$ and $\hat{z}$ are independent, so we can equate coefficients of these separately:

$$\begin{aligned} kE_{zc} &= -\omega B_{yc} \\ -kE_{yc} &= -\omega B_{zc} \end{aligned} \tag{33}$$

That is, if we pick E_{yc}, then $B_{zc} = \frac{k}{\omega}E_{yc}$ and vice versa.

We still have one more equation,

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{34}$$

but it doesn't give any new information (you can check this yourself if you want).

We see now that we actually weren't as simple as we could have been! We can pick $E_{yc} = 0$ and still get a perfectly fine wave,

$$\begin{aligned} \vec{E} &= \hat{z}E_{zc}\cos(kx - \omega t) \\ \vec{B} &= \hat{y}B_{yc}\cos(kx - \omega t) \\ B_{yc} &= -\tfrac{k}{\omega}E_{zc} \end{aligned} \tag{35}$$

with everything else zero, or we could take

$$\begin{aligned} \vec{E} &= \hat{y} E_{yc} \cos(kx - \omega t) \\ \vec{B} &= \hat{z} B_{zc} \cos(kx - \omega t) \\ B_{zc} &= \tfrac{k}{\omega} E_{yc} \end{aligned} \tag{36}$$

with everything else zero. Or, we can add them together. If we add them together, we would get a vector $\vec{E}$ pointing in the direction

$$\vec{E} = \hat{y} E_{yc} \cos(kx - \omega t) + \hat{z} E_{zc} \cos(kx - \omega t) \tag{37}$$

and the vector $\vec{B}$ would be pointing in the direction

$$\vec{B} = \hat{y} B_{yc} \cos(kx - \omega t) + \hat{z} B_{zc} \cos(kx - \omega t) \tag{38}$$

which would be perpendicular to it. (You can verify this by taking

$$\begin{aligned} \vec{E} \cdot \vec{B} &= E_x B_x + E_y B_y + E_z B_z \\ &= 0 + E_{yc} B_{yc} \cos^2(kx - \omega t) + E_{zc} B_{zc} \cos^2(kx - \omega t) \\ &= -E_{yc} \tfrac{k}{\omega} E_{zc} \cos^2(kx - \omega t) + E_{zc} \tfrac{k}{\omega} E_{yc} \cos^2(kx - \omega t) \\ &= 0 \end{aligned} \tag{39}$$

Basically, we could think of this as just rotating our coordinate system so that $\vec{E}$ lies along only the $\hat{y}$ or $\hat{z}$ axis, and then we would be back to one of the cases above.

Another way of saying this is that we could rewrite this:

$$\begin{aligned} \vec{E} &= \cos(kx - \omega t)[\hat{y} E_{yc} + \hat{z} E_{zc}] \\ \vec{B} &= \cos(kx - \omega t)[\hat{y} B_{yc} + \hat{z} B_{zc}] \\ &= \cos(kx - \omega t)[\tfrac{k}{\omega}][-\hat{y} E_{zc} + \hat{z} E_{yc}] \end{aligned} \tag{40}$$

so that we see that they are proportional to each other, with $\vec{E}$ pointing in the direction

$$\hat{y} E_{yc} + \hat{z} E_{zc} \tag{41}$$

and $\vec{B}$ pointing in the direction

$$-\hat{y} E_{zc} + \hat{z} E_{yc}\ . \tag{42}$$

If we didn't have a fixed coordinate system for any other reason, we could define a new coordinate system $\tilde{y}, \tilde{z}$ with

$$\hat{y} E_{yc} + \hat{z} E_{zc} \tag{43}$$

pointed in e.g. the $\hat{\tilde{y}}$ direction and $\hat{\tilde{z}}$ perpendicular to it.

We cannot set $\vec{E}$ all to zero and have $\vec{B}$ nonzero or $\vec{B}$ all to zero and have $\vec{E}$ nonzero and still get a solution to the equations, given E we know what B is and vice versa.

Summary:

A solution to Maxwell's equations is a transverse wave, fluctuating perpendicular to its direction of motion. Maxwell's equations relate the component of the electric field to the magnetic field and give that they are perpendicular both to each other and the direction of motion. That is, $\vec{E}$ and $\vec{B}$ are perpendicular, with $|B| = \frac{k}{\omega}|E|$. So, given E you can calculate B and vice versa (e.g. for a wave travelling in the x direction, you can use equations 36 and 35.)

You can see an applet of this at http://www.phy.ntnu.edu.tw/java/emWave/emWave.html .