EM waves bottom line and some examples for Jan. 27, 2003, Physics 7c

1. Bottom line:

- Maxwell's equations in vacuum can be combined to give the wave equation, so electromagnetic fields in vacuum are waves. They travel with the speed of light (in vacuum this is $c = 3.00\times 10^8$ m s^{-1}, vacuum meaning that the space it is travelling through has permittivity ϵ_0 and permeability μ_0, and there are no sources.
- Using $\vec{\nabla}\cdot\vec{E} = 0$ and $\vec{\nabla}\cdot\vec{B} = 0$ we see that the waves are transverse, i.e. the $\vec{E}$ and $\vec{B}$ fields vary perpendicularly to the direction of motion.
- Using

$$\begin{aligned} \vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{B} &= \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \end{aligned} \tag{1}$$

  we find that $\vec{E}$ and $\vec{B}$ are perpendicular to each other as well, and that

$$|B| = \frac{|E|}{c} \tag{2}$$

So, given $\vec{E}$ you can get $\vec{B}$ using the above, and vice versa. (You can check your results by noting that $\vec{E}\times\vec{B}$ must go in the direction of motion.) We know the relation between the directions and magnitudes of the two vectors and the direction of motion and that is enough to tell us all about one if we get the other.

2. Right hand rule

The way you take cross products of vectors is to take cross products of the components, so you need to remember

$$\begin{aligned} \hat{x}\times\hat{y} &= \hat{z} = -\hat{y}\times\hat{x} \\ \hat{y}\times\hat{z} &= \hat{x} = -\hat{z}\times\hat{y} \\ \hat{z}\times\hat{x} &= \hat{y} = -\hat{x}\times\hat{z} \end{aligned} \tag{3}$$

You can just permute the indices cyclically to remember this.

3. Examples

Here are some examples to get some practice:

- Consider

$$\vec{E} = \hat{y} E_0 \cos(kx - \omega t) \tag{4}$$

then we have

$$\vec{B} = \hat{z} \frac{E_0}{c} \cos(kx - \omega t) \tag{5}$$

- Consider

$$\vec{E} = \hat{x} E_0 \cos(kz - \omega t) \tag{6}$$

This wave is going in the z direction. You can verify that $\vec{\nabla} \cdot \vec{E} = 0$. Then we have

$$\vec{B} = \hat{y} \frac{E_0}{c} \cos(kz - \omega t) \tag{7}$$

- Consider

$$\vec{E} = \hat{x} E_0 \cos(kz + \omega t) \tag{8}$$

This wave is going in the $-z$ direction, i.e. the opposite direction from above. You can verify that $\vec{\nabla} \cdot \vec{E} = 0$ still. Then we have

$$\vec{B} = -\hat{y} \frac{E_0}{c} \cos(kz + \omega t) \tag{9}$$

- We can also have a sine wave, or any combination

$$\vec{E} = \hat{y}(E_0 \cos(kx - \omega t) + E_1 \sin(kx - \omega t)) \tag{10}$$

in which case we read off again

$$\vec{B} = \hat{z}(\frac{E_0}{c} \cos(kx - \omega t) + \frac{E_1}{c} \sin(kx - \omega t)) \tag{11}$$

Aside: Note this can be rewritten as a cosine plus a phase. Define E_2 and ϕ so that they obey:

$$\begin{aligned}&(E_0 \cos(kx - \omega t) + E_1 \sin(kx - \omega t)) = E_2 \cos\phi \cos(kx - \omega t) - E_2 \sin\phi \sin(kx - \omega t)\\ &= E_2 \cos(kx - \omega t + \phi)\end{aligned} \tag{12}$$

That is, we've defined

$$\begin{aligned}&E_2 \cos\phi = E_0 \;\text{ and }\; - E_2 \sin\phi = E_1\\ &\text{so } E_2 = E_0^2 + E_1^2\\ &-\tan\phi = E_1/E_0\end{aligned} \tag{13}$$

In general, we can write our solutions to the wave equations as cosines (with some phase(s)), which is why we have been using only cosines in class.

- The complicated combination I mentioned in lecture was

$$\vec{E} = \hat{x} E_{xc} \cos(kz - \omega t) + \hat{y} E_{yc} \sin(kz - \omega t) \tag{14}$$

which leads to a $\vec{B}$ field

$$\vec{B} = \hat{y} \frac{E_{xc}}{c} \cos(kz - \omega t) - \hat{x} \frac{E_{yc}}{c} \sin(kz - \omega t) \tag{15}$$

you can do this by plugging into Maxwell's equations, or by knowing that $\vec{E}$ and $\vec{B}$ must be perpendicular and their cross product must give the direction of motion, i.e. the $\hat{z}$ direction. If you sit at a point e.g. $z = 0$ and look at the fields, you see that

$$\vec{E}(z = 0) = \hat{x} E_{xc} \cos(\omega t) - \hat{y} E_{yc} \sin(\omega t) \tag{16}$$

which leads to a $\vec{B}(z = 0)$ field

$$\vec{B}(z = 0) = \hat{y} \frac{E_{xc}}{c} \cos(\omega t) + \hat{x} \frac{E_{yc}}{c} \sin(\omega t) \tag{17}$$

Consider these at $\omega t = 0, \pi/2, \pi, 3\pi/2$, and you see that the $\vec{E}$ and $\vec{B}$ fields are rotating around at any given point, always in the plane perpendicular to the motion. Don't worry too much about this example!

- Another case is to start with $\vec{B}$.

$$\vec{B} = \hat{y} B_0 \cos(kz - \omega t) \tag{18}$$

in which case we know that

$$\vec{E} = \hat{x} c B_0 \cos(kz - \omega t) \tag{19}$$