Physics 7C, Spring 2003
Notes on special relativity

1 Basic Idea

Everything comes from assuming the two principles of special relativity,

- The basic laws of physics are the same in all inertial frames. Any frame moving with constant velocity with respect to an inertial frame is also an inertial frame, and you can't do any experiment in any frame to tell you which frame is "really moving."
- Light propagates throuh empty space with a definite speed c independent of the speed of the source or observer.

Note that you can measure your relative velocity to another inertial frame, i.e. you can say if they are moving with velocity v relative to you, but you can't say whether they are moving, or you are moving or you both are moving–no experiment can distinguish these.

We then saw that these two assumptions led to time dilation and Lorentz contration: moving clocks run slowly and the length of an object moving relative to you is shortened in the direction of motion. In both cases, the effect depended on $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, in that lengths were shortened by $1/\gamma$ and time was less for a moving clock by a factor of $1/\gamma$, when v was taken to be the relative velocity of the person doing the measurement and the moving clock and/or length.

2 Transformations between coordinates

Systematically, if the coordinate system x', ct' is moving with relative velocity v to a coordinate system x, ct, we have

$$\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma(t' + \tfrac{v}{c^2}x') \end{aligned} \tag{1}$$

where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{2}$$

and $\gamma > 1$. We can rewrite this to look like a "rotation" in a funny space,

$$\begin{aligned} x &= x' \cosh\alpha + ct' \sinh\alpha \\ y &= y' \\ z &= z' \\ ct &= x' \sinh\alpha + ct' \cosh\alpha \end{aligned} \tag{3}$$

where

$$\begin{array}{l} \cosh\alpha = \gamma \\ \sinh\alpha = \frac{v}{c}\gamma \\ \tanh\alpha = \frac{v}{c} \\ \cosh^2\alpha - \sinh^2\alpha = 1 \end{array} \tag{4}$$

These transformations preserve the "$distance^2$" between two different positions in space and time.

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \tag{5}$$

That is, if the person in the unprimed frame considers a point x_1, ct_1 and another point x_2, ct_2, i.e. two different positions each at a different time (say y, z unchanged), and then defines

$$\begin{array}{l} \Delta x = x_2 - x_1 \\ c\Delta t = ct_2 - ct_1 \end{array} \tag{6}$$

and then calculates $(\Delta s)^2$, and the person in the primed frame takes his/her coordinates for those same points (which will be different numbers, i.e. x'_1, ct'_1 and x'_2, ct'_2) and their own $c\Delta t'$ and $\Delta x'$), they will find

$$\begin{aligned} (\Delta s)^2 &= (c\Delta t)^2 - (\Delta x)^2 \\ &= (c\Delta t')^2 - (\Delta x')^2 \end{aligned} \tag{7}$$

That is, all inertial observers measure the same $(\Delta s)^2$.

If you have two segments and want the distance along the sum of the segments, you add the quantity with units of length, that is $\Delta s = \sqrt{(\Delta s)^2}$.

The other interesting thing about $(\Delta s)^2$, the "distance (squared) in spacetime" between two points, is that if you go to a coordinate system where the two points are at the same position but different times, you get $(\Delta s)^2 = (c\Delta t)^2$, that is Δs is the time elapsed for you if you are in the coordinate system where the two points are at the same (x, y, z) positions. This time is called the "proper" time between the two points.

3 Spacetime

As space and time get all mixed up in these transformations, and we can even write them as a funny "rotation," with coshes and sinhes rather than cosines and sines, people consider space and time together to give coordinates for things, i.e. "spacetime". Spacetime has 4 dimensions: ct, and (x, y, z), but as inertial frames only have mixing between time and space in the direction of motion, people often just plot the x, ct space, and call x the direction of motion for convenience. Also, it is hard to plot more than 2 dimensions!

The distances along the x', ct' axes are shorter than along the x, ct axes, that is, $ct' = 0, x' = 1$ unit for instance will correspond to $x = 1/\gamma$ units (and ct nonzero), (Lorentz contraction), likewise $ct' = 1$ unit (and $x' = 0$) will be at a ct value of $ct = 1/\gamma$ units (and some nonzero x) (time dilation).

To read off the time or position that either person sees, you draw lines parallel to the x', ct' axes (for what x', ct' says the coordinates are) or parallel to the x, ct axes (for what x, ct says the coordinates are). Note that a line at 45 degrees (a light ray) has both $x = ct$ and $x' = ct'$ by this rule, which is good, as then for a light ray $(\Delta s)^2 = 0$ in any system, which is true.

4 "Paradoxes"

Many paradoxes appear in special relativity that are actually related to us making assumptions without realizing it. The most insidious one is that we assume we agree with someone else about what "at the same time" means. Simulatenity is relative though, in different frames. I.e. if you say two things happen at the same time and different places, someone moving with some velocity relative to you will not say that they happen at the same time. You can see this by drawing lines of constant ct and constant ct'. As these aren't

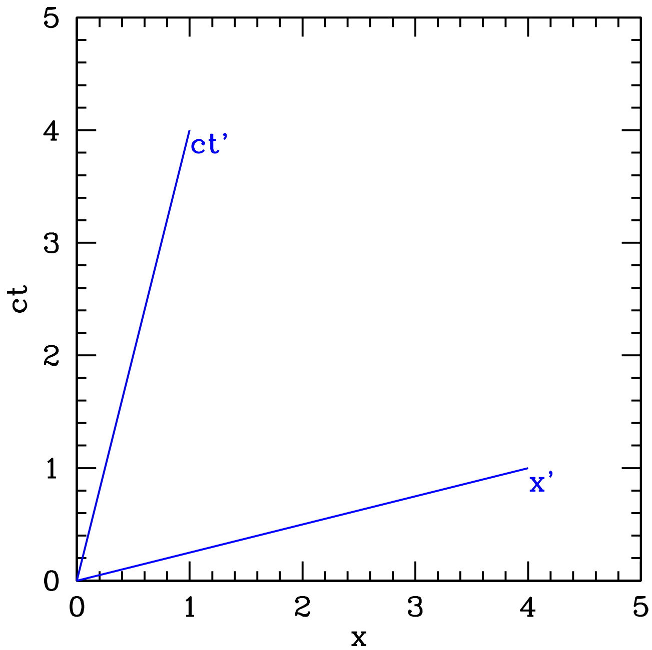

Figure 1: The relation between x and ct coordinate systems and x', ct' coordinate systems. The angle between the ct and ct' axes is θ, where $\tan\theta = v/c = \tanh\alpha$. This is also the angle between the x and x' axes. A light ray will be at 45 degrees from the x, ct axes.

parallel, two things at the same time in the x, ct frame aren't at the same time in the x', ct' frame.

The pole in the barn paradox is this sort of paradox. Someone takes a pole which has the same length as a barn when both are at rest, and then runs towards the barn with it. Then we use that something moving relative to you is shortened in the direction of motion. For the runner, the barn is shrinking and the pole won't fit. For the barn, the pole is shrinking, so of course it fits. But to say the pole fits in the barn, you want both ends of the pole in the barn *at the same time*. And you mean, at the same time according to whom? And so you find that the two disagree about whether the pole fits or not, but that they are both right. There are pictures at http://socrates.berkeley.edu/~phy7c/pb.html .

Another "paradox" is the twin paradox. In this case, two twins start out with clocks, and one goes out for a time $t_1/2$ at velocity v and then turns around and comes back with velocity v for time $t_1/2$. The other twin stays put for time t_1. Each twin can say that the other twin is moving relative to them, so that the other person should have seen less time elapsed. But then they meet and put both clocks together and ask what happens and at this point you wonder which one was right?

Here the point is that one of the twins has accelerated, when s/he turned around and came back. If you look at the paths they take on a spacetime diagram, one of them has a path with a "kink" in it. We can calculate $(\Delta s)^2$ for both paths and they are different. Then, assuming that the twin who accelerates (i.e. slows down and turns around) does so instantaneously (hard to do, but easier to calculate), we can say that except for that moment, their frame is also inertial. So we can calculate $(\Delta s)^2$ for both paths, and then interpret them as the time elapsed for the person on each path.

We use the coordinate system of the twin who stayed put to calculate.

Person 1:

$$\begin{aligned} c\Delta t &= ct_1 \\ \Delta x &= 0 \\ (\Delta s_1)^2 &= c^2 t_1^2 \end{aligned} \tag{8}$$

Person 2: We need to calculate $(\Delta s)^2$ for each part of the path separately, as each path describes inertial motion, and then add Δs for each.

On trip out:

$$\begin{aligned} c\Delta t &= ct_1/2 \\ \Delta x &= vt_1/2 \\ (\Delta s_2)^2 &= c^2 t_1^2/4 - v^2 t_1^2/4 \end{aligned} \tag{9}$$

The quantity with units of length is

$$\Delta s_2 = \sqrt{(\Delta s)^2} = ct_1 \frac{1}{2}\sqrt{1 - \frac{v^2}{c^2}} \tag{10}$$

On the trip back, you get the same thing.

Just like in normal space, to get the total length one adds two lengths, here one does the same thing. In this case, the *length* is $\Delta s = \sqrt{(\Delta s)^2}$.

So we get (note correction!)

$$(\Delta s_2) = ct_1\sqrt{1 - \frac{v^2}{c^2}} \tag{11}$$

Intepreting Δs along each path as the time taken if you are at rest in that coordinate system, we see that the travelling twin has seen less time passing.