Spin Notes, Physics 7C

Consider an electron with spin up in the z direction, that is, its spin wavefunction is

$$\psi_{spin} = \psi_{z\uparrow} \tag{1}$$

so that its angular momentum is $\frac{\hbar}{2}$.

Consider looking at an axis rotated by an angle θ from the z axis. In that direction, you will again always measure either spin $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$. This is true of a measurement of spin in any direction.

However, there is an effect from knowing that you started with $\psi_{z\uparrow}$ to start with. Call a state with spin up in the θ direction $\psi_{\theta\uparrow}$ and spin down $\psi_{\theta\downarrow}$ and normalize: $|\psi_{\theta\uparrow}|^2 = |\psi_{\theta\downarrow}|^2 = |\psi_{z\uparrow}|^2 = 1$. One can calculate (using methods we haven't covered in this class) that

$$\psi_{z\uparrow} = \cos\frac{\theta}{2}\psi_{\theta\uparrow} - \sin\frac{\theta}{2}\psi_{\theta\downarrow}\ . \tag{2}$$

(Also $\psi_{z\downarrow} = \sin\frac{\theta}{2}\psi_{\theta\uparrow} + \cos\frac{\theta}{2}\psi_{\theta\downarrow}$ but we won't use it here.) So, if you start with spin up in the z direction, then in the θ direction the probability to get spin up is $\cos^2\frac{\theta}{2}$ and to get spin down is $\sin^2\frac{\theta}{2}$. You can also consider the average spin in the θ direction if you measured many different systems all which had spin up in the z direction:

$$< spin >_{\theta\ direction} = \frac{\hbar}{2}Prob\uparrow - \frac{\hbar}{2}Prob\downarrow = \frac{\hbar}{2}[\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}] = \frac{\hbar}{2}\cos\theta \tag{3}$$

This is exactly what you'd get if spin were a classical vector with length $\frac{\hbar}{2}$ in the z direction, its projection at an angle θ would be $\frac{\hbar}{2}\cos\theta$. But to repeat, this is true only for the average value, for individual measurements you'd get $\pm\frac{\hbar}{2}$ (one or the other).

Once you have made the measurement in the θ direction, you will find either $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$, and then your wave function will *be* either $\psi_{\theta\uparrow}$ or $\psi_{\theta\downarrow}$ respectively. if you measure in a third direction after this with some other angle, and you want probabilities for that third direction, you need to then write $\psi_{\theta\uparrow}$ or $\psi_{\theta\downarrow}$ in terms of ψ's in the new direction. You can of course just rename what you call z, call θ the variable z instead, and the new angle you can call θ and then repeat what we just did, but for the new direction. For example you could then measure spin in the z direction again, but as you started with a state with definite spin in the θ direction, you will not have a definite value of spin in the z direction anymore. It is important to remember that once you measure the spin, you have a wavefunction with definite spin in that direction.

You can also see that for some angles you will end up with something definite. If you rotate by $\theta = \pi$, then you'll get only spin down, because you've just flipped the z axis. Curiously, if you rotate by $\theta = 2\pi$ you actually don't come back to exactly the same wavefunction but one with an overall minus sign. This eventually has many important consequences for the behavior of spin 1/2 (or 3/2 etc) particles and can be tied to the Pauli exclusion principle which we'll discuss soon. (see e.g. http://www.umsl.edu/~fraundor/p231/spinexcl.html).